Probability Review

CSCI 447/547 MACHINE LEARNING

# Outline

- Random Variables

- Joint and Marginal Distributions

- Conditional Distribution

- Product Rule, Chain Rule, Bayes' Rule

- Independence

Random Variables

- A random variable is some aspect of the world about which we (may) have uncertainty

  - R = Is it raining?
  - T = Is it hot or cold?
  - D = How long will it take to drive to work?

- We denote random variables with capital letters

- Random variables have domains

  - R in {true, false} (often write as {+r, -r})
 - T in {hot, cold}
  - D in [o, ∞)

Definitions
Random Variables

Random Variables

Discrete Random Variable

Continuous Random Variable
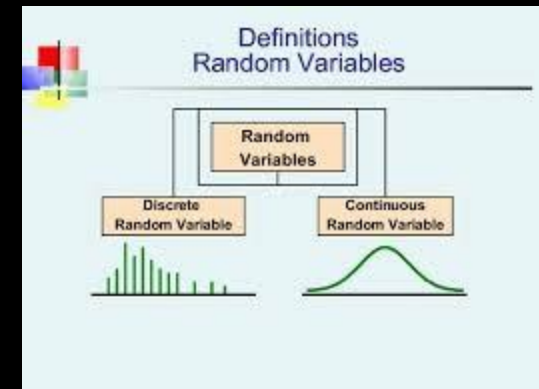
Probability Distributions

- Associate a probability with each value, or for continuous variables, a range of values

Weather:

$P(W)$

W	P
sun	0.6
rain	0.1
fog	0.3
meteor	0.0

Definitions
Random Variables

Random Variables

Discrete
Random Variable

Continuous
Random Variable
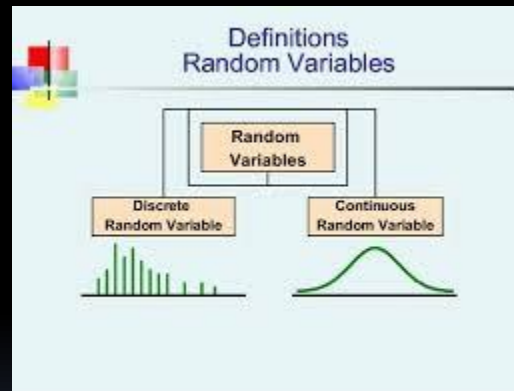
Probability Density Function

$$F(x) = P(a \leq x \leq b) = \int_a^b f(x)dx \geq 0$$

Probability Distributions

- Unobserved random variables have distributions

$P(T)$

T	P
hot	0.5
cold	0.5

$P(W)$

W	P
sun	0.6
rain	0.1
fog	0.3
meteor	0.0

- A distribution is a TABLE of probabilities of values

- A probability (lower case value) is a single number $P(W = rain) = 0.1$

- Must have: $\forall x \ P(X = x) \geq 0$ and $\sum_{x} P(X = x) = 1$

Joint Distributions

- A *joint distribution* over a set of random variables: $X_1, X_2, \ldots X_n$ specifies a real number for each assignment (or *outcome*):

$$P(X_1 = x_1, X_2 = x_2, \ldots X_n = x_n)$$

$$P(x_1, x_2, \ldots x_n)$$

$P(T,W)$

T	W	P
hot	sun	0.4
hot	rain	0.1
cold	sun	0.2
cold	rain	0.3

- Must obey: $P(x_1, x_2, \ldots x_n) \geq 0$

$$\sum_{(x_1, x_2, \ldots x_n)} P(x_1, x_2, \ldots x_n) = 1$$

- Size of distribution if n variables with domain sizes d?

  - For all but the smallest distributions, impractical to write out!

Probabilistic Models

- A probabilistic model is a joint distribution over a set of random variables

- Probabilistic models:
  - (Random) variables with domains
  - Assignments are called *outcomes*
  - Joint distributions: say whether assignments (outcomes) are likely
  - *Normalized:* sum to 1.0
  - Ideally: only certain variables directly interact

Distribution over T,W

T	W	P
hot	sun	0.4
hot	rain	0.1
cold	sun	0.2
cold	rain	0.3

Events

- An *event* is a set E of outcomes

$$P(E) = \sum_{(x_1 \ldots x_n) \in E} P(x_1 \ldots x_n)$$

- From a joint distribution, we can calculate the probability of any event

  $P(T, W)$

  - Probability that it's hot AND sunny?

  - Probability that it's hot?
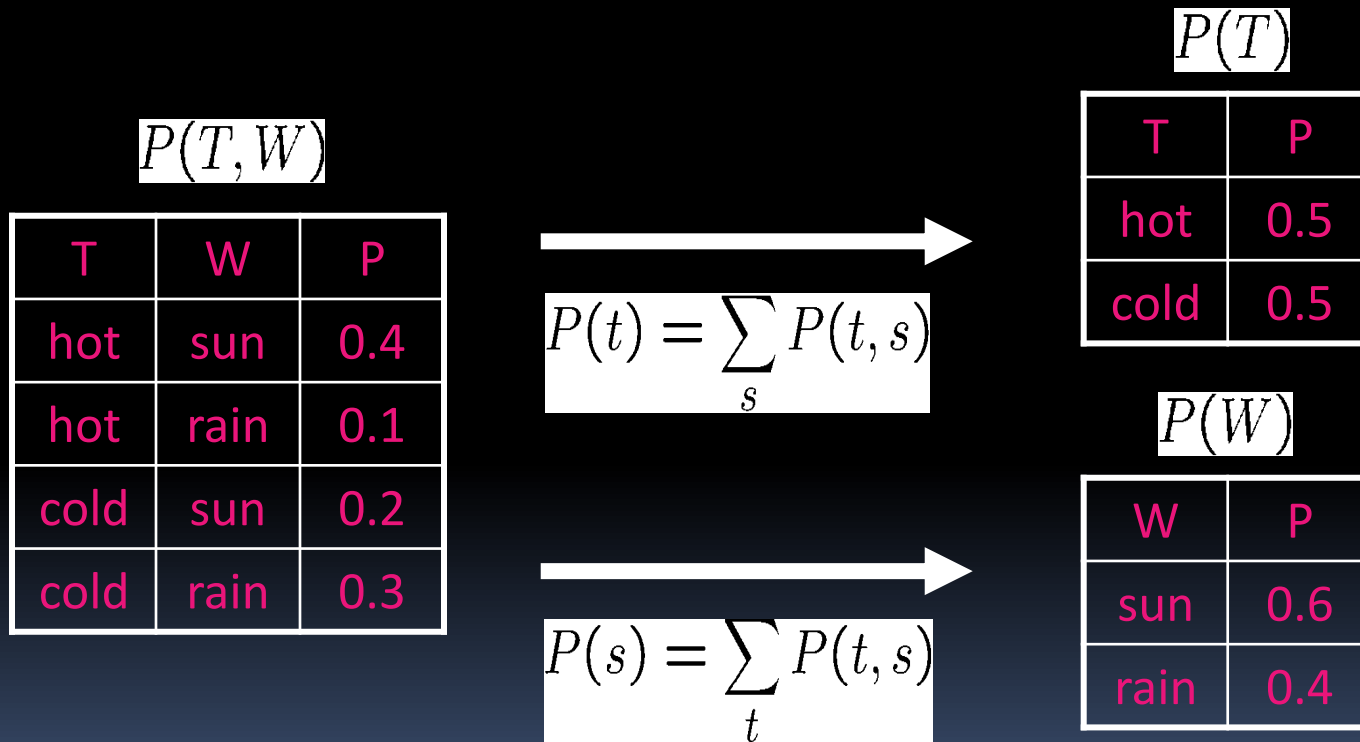
  - Probability that it's hot OR sunny?

- Often the events we care about are *partial assignments*, like P(T=hot)

T	W	P
hot	sun	0.4
hot	rain	0.1
cold	sun	0.2
cold	rain	0.3

Marginal Distributions

- Marginal distributions are sub-tables which eliminate variables
- Marginalization (summing out): Combine collapsed rows by adding

$P(T,W)$

T	W	P
hot	sun	0.4
hot	rain	0.1
cold	sun	0.2
cold	rain	0.3

$$P(t) = \sum_s P(t,s)$$

$$P(s) = \sum_t P(t,s)$$

$P(T)$

T	P
hot	0.5
cold	0.5

$P(W)$

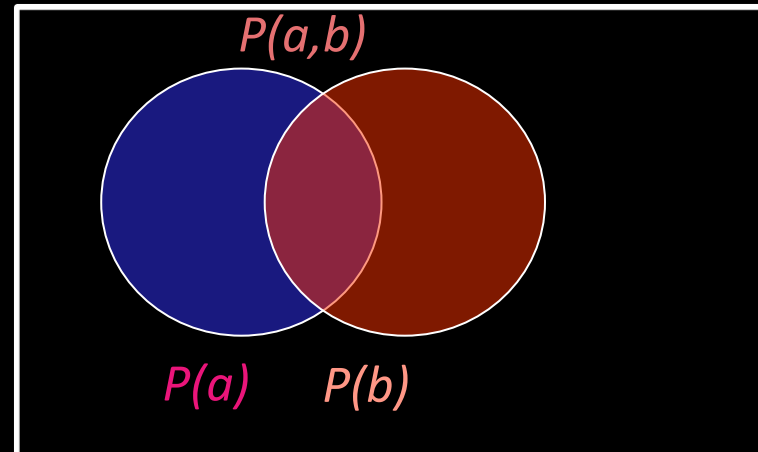| W	P
sun	0.6
rain	0.4

$$P(X_1 = x_1) = \sum_{x_2} P(X_1 = x_1, X_2 = x_2)$$

Conditional Probabilities

- A simple relation between joint and conditional probabilities
  - In fact, this is taken as the *definition* of a conditional probability

$$P(a|b) = \frac{P(a,b)}{P(b)}$$

$P(a,b)$

$P(a)$ $P(b)$

$P(T,W)$

T	W	P
hot	sun	0.4
hot	rain	0.1
cold	sun	0.2
cold	rain	0.3

$$P(W = s|T = c) = \frac{P(W = s, T = c)}{P(T = c)} = \frac{0.2}{0.5} = 0.4$$
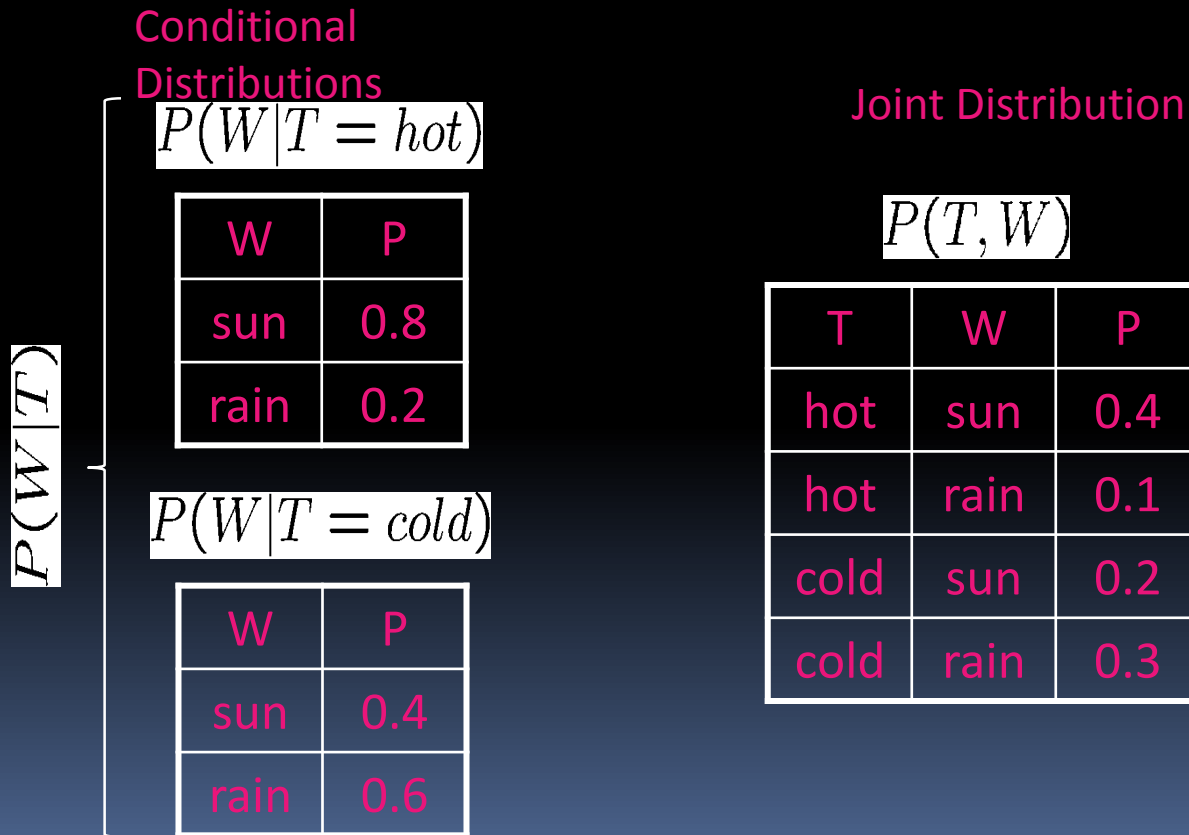
$$= P(W = s, T = c) + P(W = r, T = c)$$

$$= 0.2 + 0.3 = 0.5$$

Conditional Distributions

- Conditional distributions are probability distributions over some variables given fixed values of others

Conditional Distributions

$P(W|T = hot)$

W	P
sun	0.8
rain	0.2

$P(W|T)$

$P(W|T = cold)$

W	P
sun	0.4
rain	0.6

Joint Distribution

$P(T,W)$

T	W	P
hot	sun	0.4
hot	rain	0.1
cold	sun	0.2
cold	rain	0.3

Normalization Trick

$$= \frac{P(W = s, T = c)}{P(T = c)}$$

$$= \frac{P(W = s, T = c)}{P(W = s, T = c) + P(W = r, T = c)}$$

$$= \frac{0.2}{0.2 + 0.3} = 0.4$$

$P(T, W)$

T	W	P
hot	sun	0.4
hot	rain	0.1
cold	sun	0.2
cold	rain	0.3

$P(W|T = c)$

W	P
sun	0.4
rain	0.6

$$P(W = r|T = c) = \frac{P(W = r, T = c)}{P(T = c)}$$

$$= \frac{P(W = r, T = c)}{P(W = s, T = c) + P(W = r, T = c)}$$

$$= \frac{0.3}{0.2 + 0.3} = 0.6$$

Normalization Trick

$$= \frac{P(W = s, T = c)}{P(T = c)}$$

$$= \frac{P(W = s, T = c)}{P(W = s, T = c) + P(W = r, T = c)}$$

$$= \frac{0.2}{0.2 + 0.3} = 0.4$$

$P(T, W)$

T	W	P
hot	sun	0.4
hot	rain	0.1
cold	sun	0.2
cold	rain	0.3

SELECT the joint probabilities matching the evidence →

$P(c, W)$

T	W	P
cold	sun	0.2
cold	rain	0.3

NORMALIZE the selection (make it sum to one) →

$P(W|T = c)$

W	P
sun	0.4
rain	0.6

$$P(W = r|T = c) = \frac{P(W = r, T = c)}{P(T = c)}$$

$$= \frac{P(W = r, T = c)}{P(W = s, T = c) + P(W = r, T = c)}$$

$$= \frac{0.3}{0.2 + 0.3} = 0.6$$

To Normalize

- (Dictionary) To bring or restore to a normal condition

All entries sum to ONE

- Procedure:
    - Step 1: Compute Z = sum over all entries
    - Step 2: Divide every entry by Z

- Example 1

W	P
sun	0.2
rain	0.3

Normalize

Z = 0.5

W	P
sun	0.4
rain	0.6

- Example 2

T	W	P
hot	sun	20
hot	rain	5
cold	sun	10
cold	rain	15

Normalize

Z = 50

T	W	P
hot	sun	0.4
hot	rain	0.1
cold	sun	0.2
cold	rain	0.3

The Product Rule

- Sometimes have conditional distributions but want the joint

$$P(y)P(x|y) = P(x,y) \qquad \Longleftrightarrow \qquad P(x|y) = \frac{P(x,y)}{P(y)}$$

The Product Rule

$$P(y)P(x|y) = P(x,y)$$

- Example:

$P(W)$

R	P
sun	0.8
rain	0.2

$P(D|W)$

D	W	P
wet	sun	0.1
dry	sun	0.9
wet	rain	0.7
dry	rain	0.3

$P(D,W)$

D	W	P
wet	sun	0.08
dry	sun	0.72
wet	rain	0.14
dry	rain	0.06

The Chain Rule

- More generally, can always write any joint distribution as an incremental product of conditional distributions

$$P(x_1, x_2, x_3) = P(x_1)P(x_2|x_1)P(x_3|x_1, x_2)$$

$$P(x_1, x_2, \ldots x_n) = \prod_i P(x_i|x_1 \ldots x_{i-1})$$

Bayes' Rule

- Two ways to factor a joint distribution over two variables:

$$P(x, y) = P(x|y)P(y) = P(y|x)P(x)$$

- Dividing, we get:

$$P(x|y) = \frac{P(y|x)}{P(y)}P(x)$$

- Why is this at all helpful?

  - Lets us build one conditional from its reverse
  - Often one conditional is tricky but the other one is simple

Example with Bayes' Rule

- Example: Diagnostic probability from causal probability:

- Example:
  - M: meningitis, S: stiff neck

$$P(+m) = 0.0001$$
$$P(+s| + m) = 0.8$$
$$P(+s| - m) = 0.01$$

Example givens

$$P(+m| + s) = \frac{P(+s| + m)P(+m)}{P(+s)} = \frac{P(+s| + m)P(+m)}{P(+s| + m)P(+m) + P(+s| - m)P(-m)} = \frac{0.8 \times 0.0001}{0.8 \times 0.0001 + 0.01 \times 0.999}$$

  - Note: posterior probability of meningitis still very small
  - Note: you should still get stiff necks checked out! Why?

Independence

- Two variables are *independent* in a joint distribution if:

$$P(X,Y) = P(X)P(Y)$$

$$X \perp\!\!\!\perp Y$$

$$\forall x, y \, P(x,y) = P(x)P(y)$$

  - Says the joint distribution *factors* into a product of two simple ones
  - Usually variables aren't independent!

- Can use independence as a *modeling assumption*
  - Independence can be a simplifying assumption
  - *Empirical* joint distributions: at best "close" to independent

Conditional Independence

- P(Toothache, Cavity, Catch)

- If I have a cavity, the probability that the probe catches in it doesn't depend on whether I have a toothache:
  - P(+catch | +toothache, +cavity) = P(+catch | +cavity)

- The same independence holds if I don't have a cavity:
  - P(+catch | +toothache, -cavity) = P(+catch| -cavity)

- Catch is *conditionally independent* of Toothache given Cavity:
  - P(Catch | Toothache, Cavity) = P(Catch | Cavity)

- Equivalent statements:
  - P(Toothache | Catch , Cavity) = P(Toothache | Cavity)
  - P(Toothache, Catch | Cavity) = P(Toothache | Cavity) P(Catch | Cavity)
  - One can be derived from the other easily

Conditional Independence

- Unconditional (absolute) independence very rare (why?)

- *Conditional independence* is our most basic and robust form of knowledge about uncertain environments.

- X is conditionally independent of Y given Z $X \perp\!\!\!\perp Y \mid Z$

  if and only if:
  $$\forall x, y, z : P(x, y \mid z) = P(x \mid z)P(y \mid z)$$

  or, equivalently, if and only if
  $$\forall x, y, z : P(x \mid z, y) = P(x \mid z)$$

Probability Recap

- Conditional probability $P(x|y) = \dfrac{P(x,y)}{P(y)}$

- Product rule $P(x,y) = P(x|y)P(y)$

- Chain rule
$$P(X_1, X_2, \ldots X_n) = P(X_1)P(X_2|X_1)P(X_3|X_1, X_2)\ldots$$
$$= \prod_{i=1}^{n} P(X_i|X_1, \ldots, X_{i-1})$$

- X, Y independent if and only if: $\forall x, y : P(x,y) = P(x)P(y)$

- X and Y are conditionally independent given Z if and only if: $\forall x, y, z : P(x,y|z) = P(x|z)P(y|z)$ $\quad X \perp\!\!\!\perp Y | Z$

Summary

- Random Variables
- Joint and Marginal Distributions
- Conditional Distribution
- Product Rule, Chain Rule, Bayes' Rule
- Independence