FRACTALS

OUTLINE

- Chaotic Systems
- Strange Attractors
- Newton-Raphson
- Diffusion Limited Aggregation
- Fractal Geometry
- L-Systems
- Iterative Function Systems (IFS)

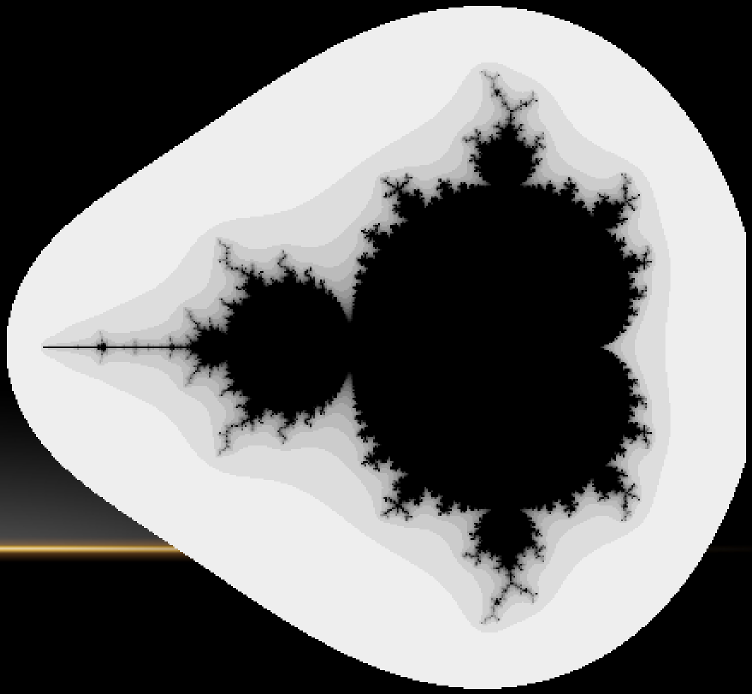
WHAT IS A FRACTAL?

- Geometric
    - A rough or fragmented geometric shape that can be subdivided in parts, each of which is (at least approximately) a reduced/size copy of the whole.
- Mathematical
    - A set of points whose fractal dimension exceeds its topological dimension.

http://paulbourke.net/fractals/fracintro/

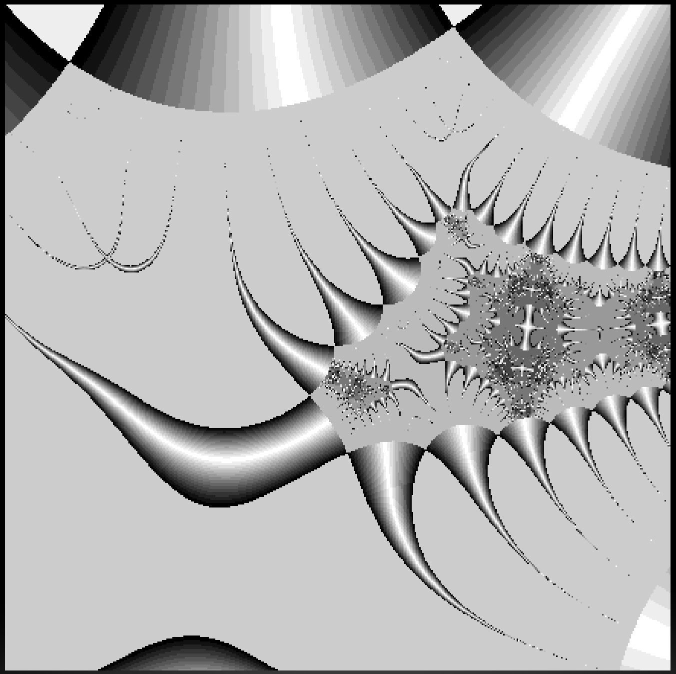
CHAOTIC SYSTEMS

- The classic Mandelbrot below has been the image that has greatly popularized chaotic and fractal systems. The Mandelbrot set is created by a general technique where a function of the form $z_{n+1} = f(z_n)$ is used to create a series of a complex variable. In the case of the Mandelbrot the function is $f(z_n) = z_n^2 + z_o$. This series in generated for every initial point z_o on some partition of the complex plane. To draw an image on a computer screen the point under consideration is colored depending on the behavior of the series which will act in one of the following ways:
  - (a) decay to 0
    (b) tend to infinity
    (c) oscillate among a number of states
    (d) exhibits no discernible pattern



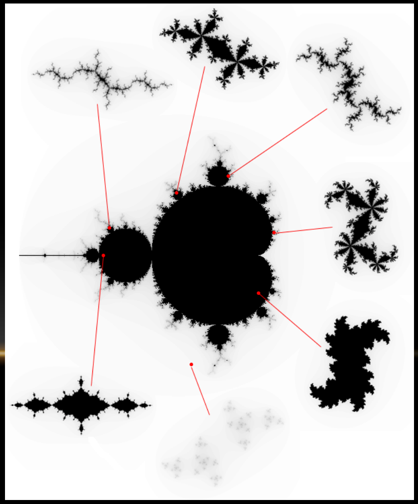
CHAOTIC SYSTEMS

- An example using the same technique but a different function is called "biomorphs" by C.A.Pickover. It uses the function $f(z_n) = \sin(z_n) + e^z + c$ and gives rise to many biological looking creatures depending on the value of the constant "c".



CHAOTIC SYSTEMS

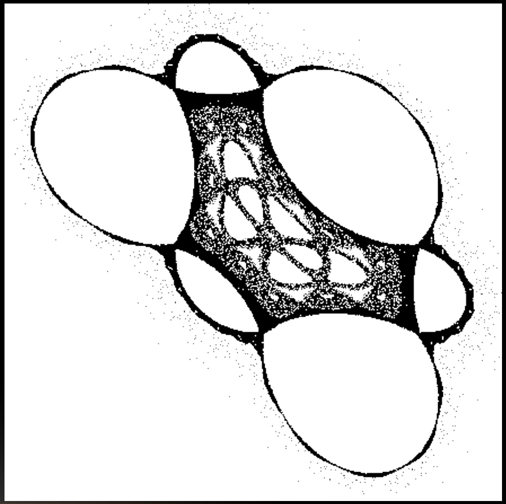
- The Julia set is named after the French mathematician Gaston Julia who investigated their properties circa 1915 and culminated in his famous paper in 1918. While the Julia set is now associated with a simpler polynomial, Julia was interested in the iterative properties of a more general expression, namely $z^4 + z^3/(z-1) + z^2/(z^3 + 4 z^2 + 5) + c$.
- The Julia set is now associated with those points $z = x + iy$ on the complex plane for which the series $z_{n+1} = z_n^2 + c$ does not tend to infinity. c is a complex constant, one gets a different Julia set for each c. The initial value z_0 for the series is each point in the image plane. In the broader sense the exact form of the iterated function may be anything, the general form being $z_{n+1} = f(z_n)$, interesting sets arise with nonlinear functions f(z). Commonly used functions include the following:
  - $z_{n+1} = c \sin(z_n)$
  - $z_{n+1} = c \exp(z_n)$
  - $z_{n+1} = c\, i \cos(z_n)$
  - $z_{n+1} = c\, z_n (1 - z_n)$



http://paulbourke.net/fractals/juliaset/

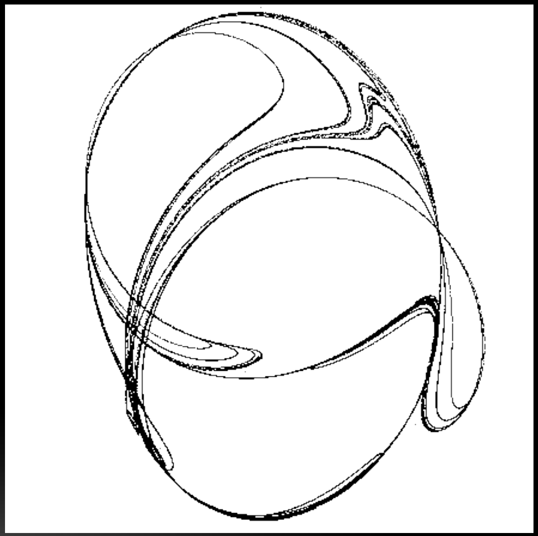
STRANGE ATTRACTORS

- A second technique, often called "hopalong" after an article in Scientific American in 1986 by Barry Martin, is normally used to represent the strange attractor of a chaotic system, for example, the well known Julia set. In this case each coordinate generated by the series is drawn as a small point, ie: we hop-along from one point to the next. For an image on a plane the series is either an equation of a complex variable or else there are two interrelated equations, one for the x and one for the y coordinate. As an example consider the following function:
  - $x_{n+1} = y_n - \text{sign}(x_n)\ |b\ x_n - c|^{½}$
  - $y_{n+1} = a - x_n$
- This series of x,y coordinates is specified by an initial point x_o, y_o and three constants a,b, and c. The following is an example where a=0.4, b=1, and c=0.

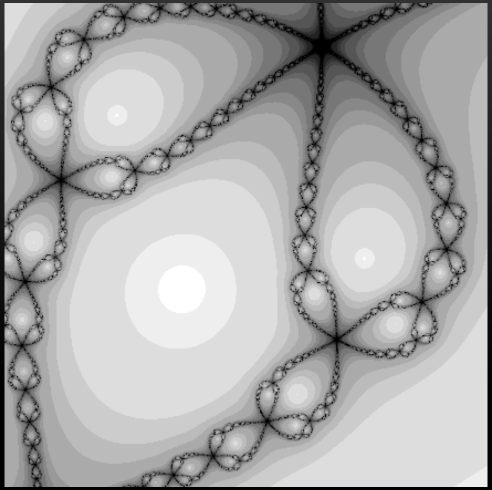


STRANGE ATTRACTORS

- Another example attributed to Peter de Jong uses the two equations:
  - $x_{n+1} = \sin(a\ y_n) - \cos(b\ x_n)$
  - $y_{n+1} = \sin(c\ x_n) - \cos(d\ y_n)$
- This gives swirling tendrils that appear three dimensional, an example is shown below where a = -2.24, b = -0.65, c = 0.43, d = -2.43.



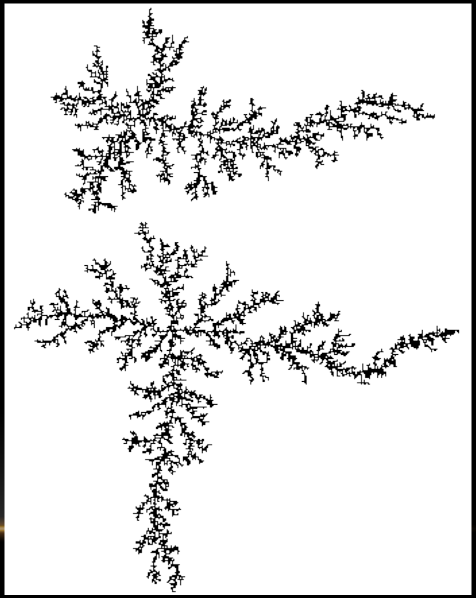
NEWTON RAPHSON



- This technique is based on the Newton Raphson method of finding the solution (roots) to a polynomial equation of the form:
  - $f(z) = a_0 + a_1 z + a_2 z^2 + \ldots + a_n z^n = 0$
- The method generates a series where the n+1'th approximation to the solution is given by:
- $z_{n+1} = z_n - \frac{f(z_n)}{f'(z_n)}$
  - where $f'(z_n)$ is the slope (first derivative) of f(z) evaluated at z_n. To create a 2D image using this technique each point in a partition of the plane is used as initial guess, z_o, to the solution. The point is colored depending on which solution is found and/or how long it took to arrive at the solution. A simple example is an application of the above to find the three roots of the polynomial z*z*z - 1 = 0. The above shows the appearance of a small portion of the positive real and imaginary quadrant of the complex plane. A trademark of chaotic systems is that very similar initial conditions can give rise to very different behaviour. In the image shown there are points very close together one of which converges to the solution very fast and the other converges very slowly.

DIFFUSION LIMITED AGGREGATION

- "The most useful fractals involve chance ... both their regularities and their irregularities are statistical." - Benoit B. Mandelbrot.
- Many attractive images can be generated using theory from areas of Chemistry and Physics. One such example is diffusion limited aggregation or DLA which describes, among other things, the diffusion and aggregation of zinc ions in an electrolytic solution onto electrodes. Another more colorful description involves a city square surrounded by taverns. Drunks leave the taverns and stagger randomly around the square until they finally trip over one of their insensate companions at which time lulled by the sounds of peaceful snoring they lie down and fall asleep. The tendril like structure is an aerial view of the sleeping crowd in the morning. .

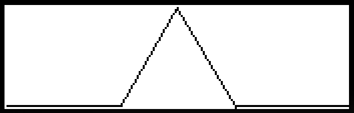


FRACTAL GEOMETRY

- A more mathematical description of dimension is based on how the "size" of an object behaves as the linear dimension increases. In one dimension consider a line segment. If the linear dimension of the line segment is doubled then obviously the length (characteristic size) of the line has doubled. In two dimensions, ff the linear dimensions of a rectangle for example is doubled then the characteristic size, the area, increases by a factor of 4. In three dimensions if the linear dimension of a box are doubled then the volume increases by a factor of 8. This relationship between dimension D, linear scaling L and the resulting increase in size S can be generalized and written as:
    - $S = L^D$
- This is just telling us mathematically what we know from everyday experience. If we scale a two dimensional object for example then the area increases by the square of the scaling. If we scale a three dimensional object the volume increases by the cube of the scale factor. Rearranging the above gives an expression for dimension depending on how the size changes as a function of linear scaling, namely:
    - $D=\frac{\log(S)}{\log(L)}$

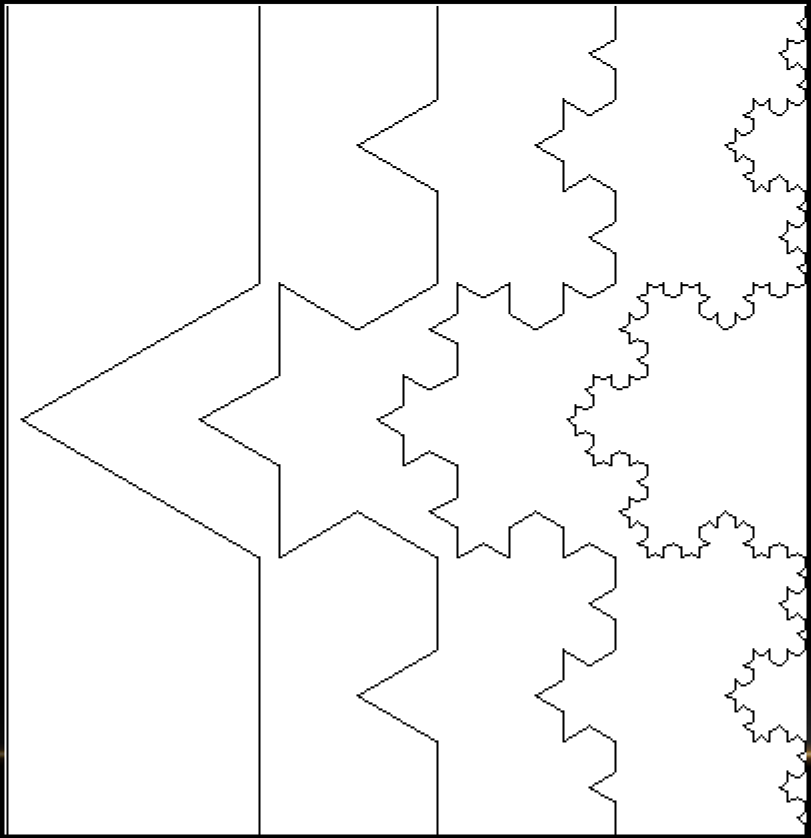
FRACTAL GEOMETRY

- In the examples above the value of D is an integer, either 1, 2, or 3, depending on the dimension of the geometry. This relationship holds for all Euclidean shapes. There are however many shapes which do not conform to the integer based idea of dimension given above in both the intuitive and mathematical descriptions. That is, there are objects which appear to be curves for example but which a point on the curve cannot be uniquely described with just one number. If the earlier scaling formulation for dimension is applied the formula does not yield an integer. There are shapes that lie in a plane but if they are linearly scaled by a factor L, the area does not increase by L squared but by some non integer amount. These geometries are called fractals! One of the simpler fractal shapes is the von Koch snowflake. The method of creating this shape is to repeatedly replace each line segment with the following 4 line segments.



FRACTAL GEOMETRY

- The process starts with a single line segment and continues for ever. The first few iterations of this procedure are shown below.

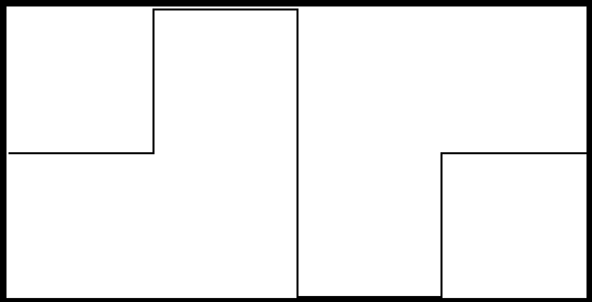


FRACTAL LANDSCAPES: L-SYSTEMS

- The following is based on L-Systems as described in "Lecture Notes in Biomathematics" by Przemyslaw Prusinkiewcz and James Hanan. A brief description of an 0L system will be presented here but for a more complete description the user should consult the literature.
- **Simpleminded example of 0L system** A string of characters (symbols) is rewritten on each iteration according to some replacement rules. Consider an initial string (axiom)
    - F+F+F+F
- and a rewriting rule
    - F --> F+F-F-FF+F+F-F
- After one iteration the following string would result
    - F+F-F-FF+F+F-F + F+F-F-FF+F+F-F + F+F-F-FF+F+F-F + F+F-F-FF+F+F-F
- For the next iteration the same rule is applied but now to the string resulting from the last iteration
    - F+ F-F-FF+ F+ F-F+ F+ F-F-FF+ F+ F-F-F+ F-F-FF+ F+ F-F-F+ F-F-FF+ F+ F-FF+ F-F-FF+ F+ F-F+ F+ F-F-FF+ F+ F-F+ F+ F-F-FF+ F+ F-F-F+ F-F-FF+ F+ F-F+ F+ F-F-FF+ F+ F-F+ F+ F-F-FF+ F+ F-F-F+ F-F-FF+ F+ F-F-F+ F-F-FF+ F+ F-FF+ F-F-FF+ F+ F-F+ F+ F-F-FF+ F+ F-F+ F+ F-F-FF+ F+ F-F-F+ F-F-FF+ F+ F-F+ F+ F-F-FF+ F+ F-F+ F+ F-F-FF+ F+ F-F-F+ F-F-FF+ F+ F-F-F+ F-F-FF+ F+ F-FF+ F-F-FF+ F+ F-F+ F+ F-F-FF+ F+ F-F+ F+ F-F-FF+ F+ F-F-F+ F-F-FF+ F+ F-F+ F+ F-F-FF+ F+ F-F+ F+ F-F-FF+ F+ F-F-F+ F-F-FF+ F+ F-F-F+ F-F-FF+ F+ F-FF+ F-F-FF+ F+ F-F+ F+ F-F-FF+ F+ F-F+ F+ F-F-FF+ F+ F-F-F+ F-F-FF+ F+ F-F

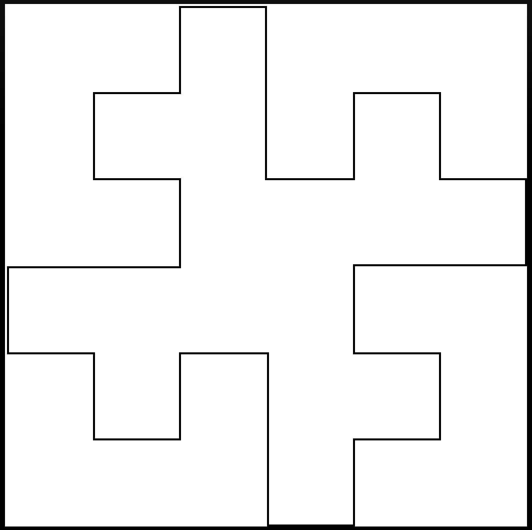
FRACTAL LANDSCAPES: L-SYSTEMS

- Some symbols are now given a graphical meaning, for example, F means move forward drawing a line, + means turn right by some predefined angle (90 degrees in this case), - means turn left. Using these symbols the initial string F+F+F+F is just a rectangle (ø = 90). The replacement rule F --> F+F-F-FF+F+F-F replaces each forward movement by the following figure

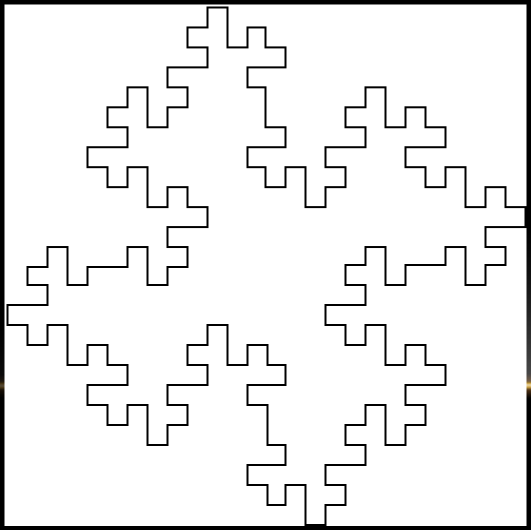


FRACTAL LANDSCAPES: L-SYSTEMS

- The first iteration interpreted graphically is

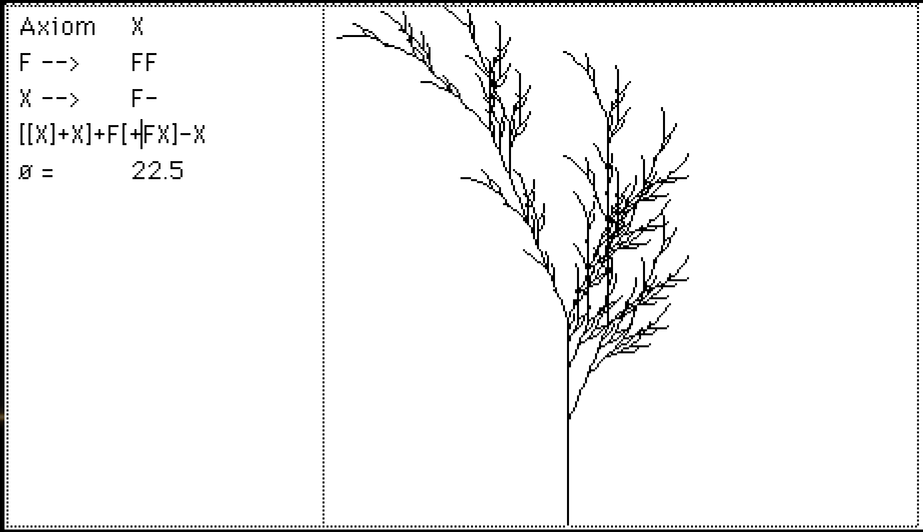


- The next iteration interpreted graphically is:



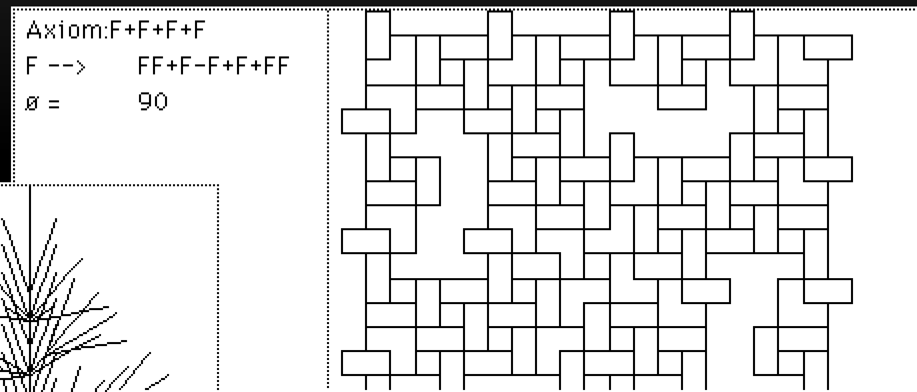
FRACTAL LANDSCAPES: L-SYSTEMS

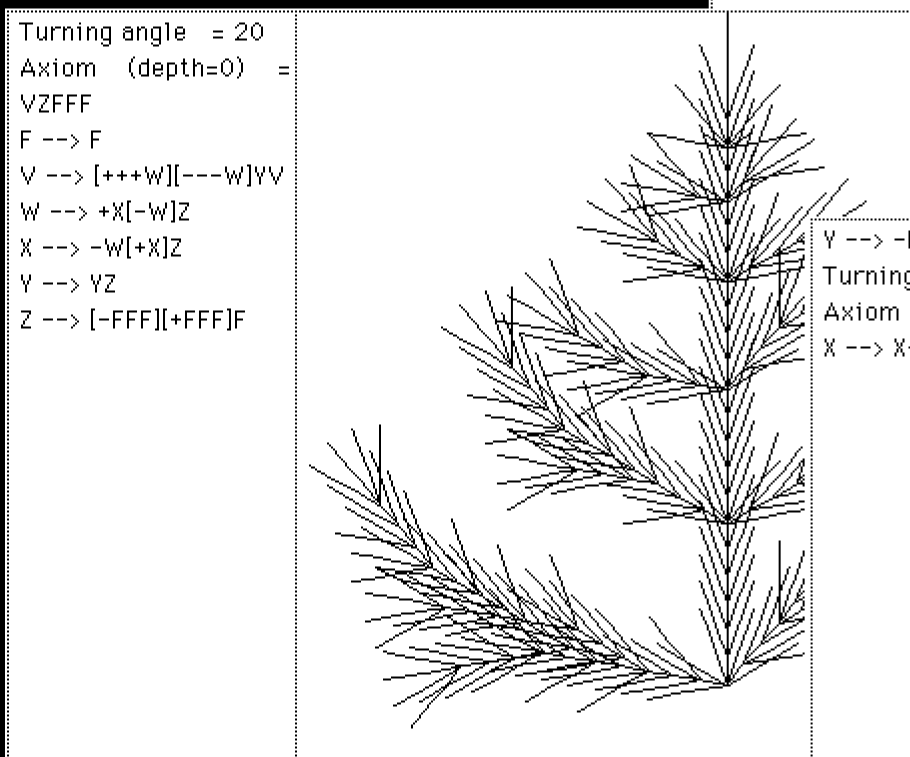
- Recent usage of L-Systems is for the creation of realistic looking objects that occur in nature and in particular the branching structure of plants. One of the important characteristics of L systems is that only a small amount of information is required to represent very complex objects. So while the bushes in figure 9 contain many thousands of lines they can be described in a database by only a few bytes of data, the actual bushes are only "grown" when required for visual presentation. Using suitably designed L-System algorithms it is possible to design the L-System production rules that will create a particular class of plant.

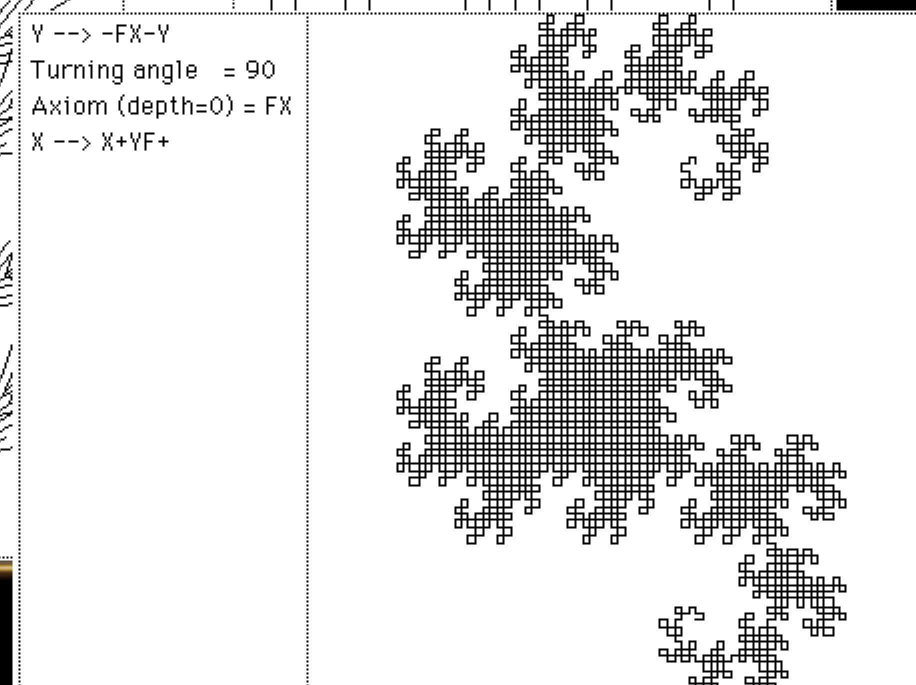


FRACTAL LANDSCAPES: L-SYSTEMS

- Further examples:







ITERATED FUNCTION SYSTEMS (IFS)

- Instead of working with lines as in L systems, IFS replaces polygons by other polygons as described by a generator. On every iteration each polygon is replaced by a suitably scaled, rotated, and translated version of the polygons in the generator. The next slide shows one such generator made of rectangles. From this geometric description it is also possible to derive a hopalong description which gives the image that would be created after iterating the geometric model to infinity. The description of this is a set of contractive transformations on a plane of the form:

$$\begin{bmatrix} x_n \\ y_n \end{bmatrix} = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \begin{bmatrix} x_{n-1} \\ y_{n-1} \end{bmatrix} + \begin{bmatrix} e \\ f \end{bmatrix}$$

- each with an assigned probability. To run the system an initial point is chosen and on each iteration one of the transformation is chosen randomly according to the assigned probabilities, the resulting points (xn,yn) are drawn on the page. As in the case of L systems, if the IFS code for a desired image can be determined (by something called the Collage theorem) then large data compression ratios can be achieved. Instead of storing the geometry of the very complex object just the IFS generator needs to be stored and the image can be generated when required. The fundamental iterative process involves replacing rectangles with a series of rectangles called the generator. The rectangles are replaced by a suitably scaled, translated, and rotated version of the generator.

ITERATED FUNCTION SYSTEMS (IFS)

IFS FERN

http://paulbourke.net/fractals/ifs_fern_a/

MORE INSPIRATION

https://www.fractalus.com/gallery/image/a-million-shades-of-green/?gallery=best;page=1

https://www.fractalus.com/gallery/about/?gallery=newest;page=1

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