GEOMETRY

OBJECTIVES

- Introduce the elements of geometry
  - Scalars
  - Vectors
  - Points
- Look at the mathematical operations among them
- Define basic primitives
  - Line segments
  - Polygons
- Look at some uses for these operations

BASIC ELEMENTS

- Geometry is the study of the relationships among objects in an n-dimensional space
  - In computer graphics, we are interested in objects that exist in three dimensions
- Want a minimum set of primitives from which we can build more sophisticated objects
- We will need three basic elements
  - Scalars
  - Vectors
  - Points

GEOMETRY

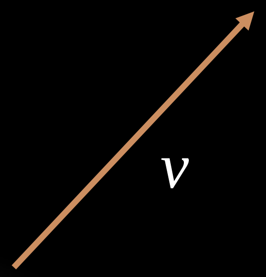
- When we learned simple geometry, most of us started with a Cartesian approach
  - Points were at locations in space **p**=(x,y,z)
  - We derived results by algebraic manipulations involving these coordinates

SCALARS

- Really need three basic elements in geometry
  - Scalars, Vectors, Points
- Scalars can be defined as members of sets which can be combined by two operations (addition and multiplication) obeying some fundamental axioms (associativity, commutivity, inverses)
- Examples include the real and complex number systems under the ordinary rules with which we are familiar
- Scalars alone have no geometric properties

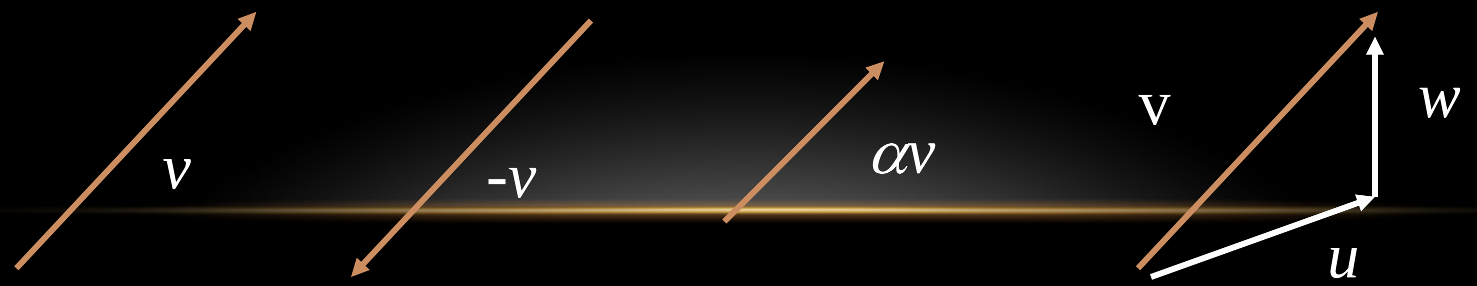
VECTORS

- Physical definition: a vector is a quantity with two attributes
  - Direction
  - Magnitude
- Examples include
  - Force
  - Velocity
  - Directed line segments
    - Most important example for graphics
    - Can map to other types



VECTOR OPERATIONS

- Every vector has an inverse
  - Same magnitude but points in opposite direction
- Every vector can be multiplied by a scalar
- There is a zero vector
  - Zero magnitude, undefined orientation
- The sum of any two vectors is a vector
  - Use head-to-tail axiom



LINEAR VECTOR SPACES

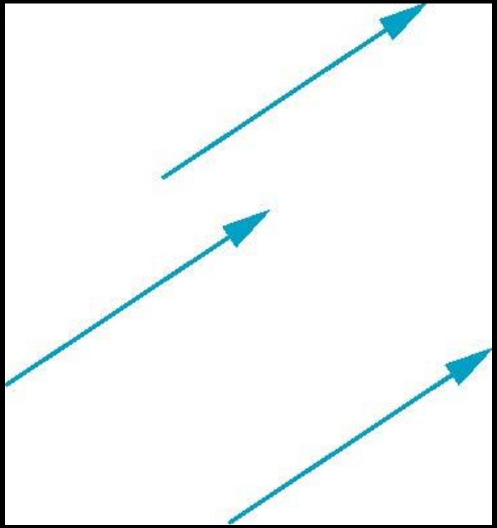
- Mathematical system for manipulating vectors
- Operations
  - Scalar-vector multiplication $u=\alpha v$
  - Vector-vector addition: $w=u+v$
- Expressions such as

  $v=u+2w-3r$

Make sense in a vector space

VECTORS LACK POSITION

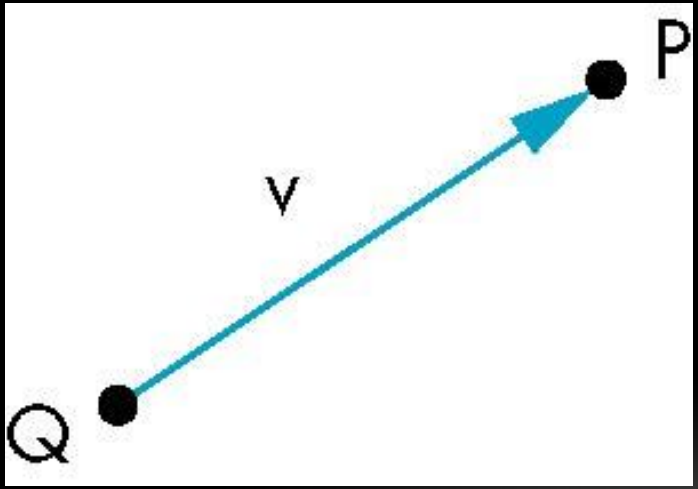
- These vectors are identical
  - Same length and magnitude



- Vectors spaces insufficient for geometry
  - Still need points

POINTS

- Location in space
- Operations allowed between points and vectors
    - Point-point subtraction yields a vector
    - Equivalent to point-vector addition



$$v = P - Q$$

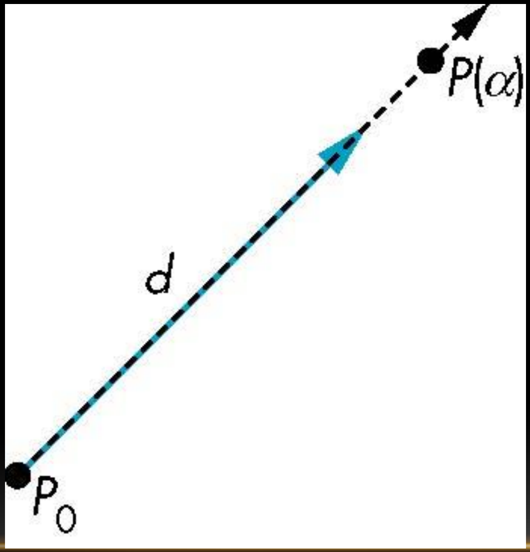
$$P = v + Q$$

AFFINE SPACES

- Point + a vector space
- Operations
  - Vector-vector addition
  - Scalar-vector multiplication
  - Point-vector addition
  - Scalar-scalar operations
- For any point define
  - $1 \bullet P = P$
  - $0 \bullet P = \mathbf{0}$ (zero vector)

LINES

- Consider all points of the form
  - $P(\alpha)=P_0 + \alpha \mathbf{d}$
  - Set of all points that pass through P_0 in the direction of the vector $\mathbf{d}$



RAYS AND LINE SEGMENTS

- If $\alpha >= 0$, then $P(\alpha)$ is the *ray* leaving P_0 in the direction $\mathbf{d}$

If we use two points to define v, then

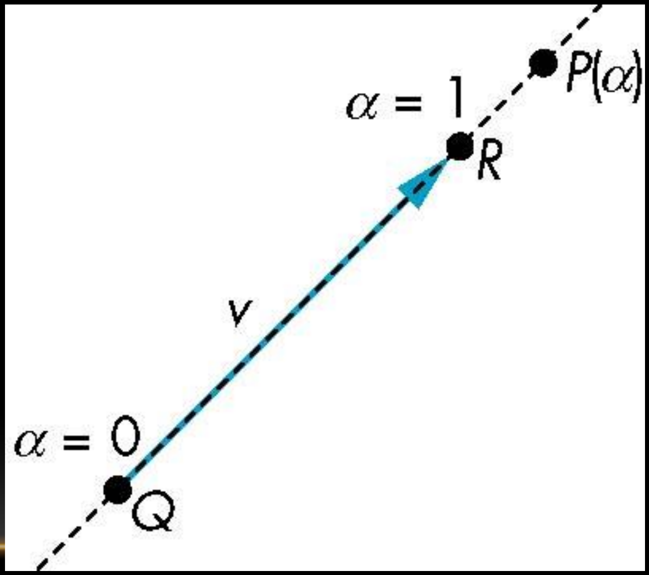
$P(\alpha) = Q + \alpha (R-Q)=Q+\alpha v$

$=\alpha R + (1-\alpha)Q$

For $0<=\alpha<=1$ we get all the

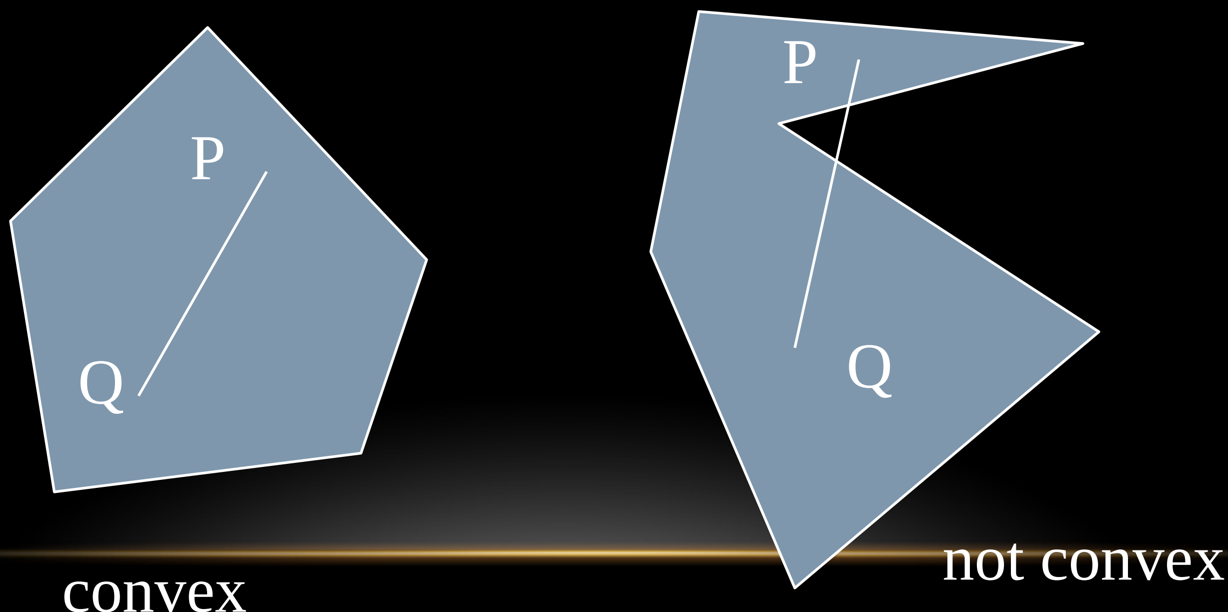
points on the *line segment*

joining R and Q



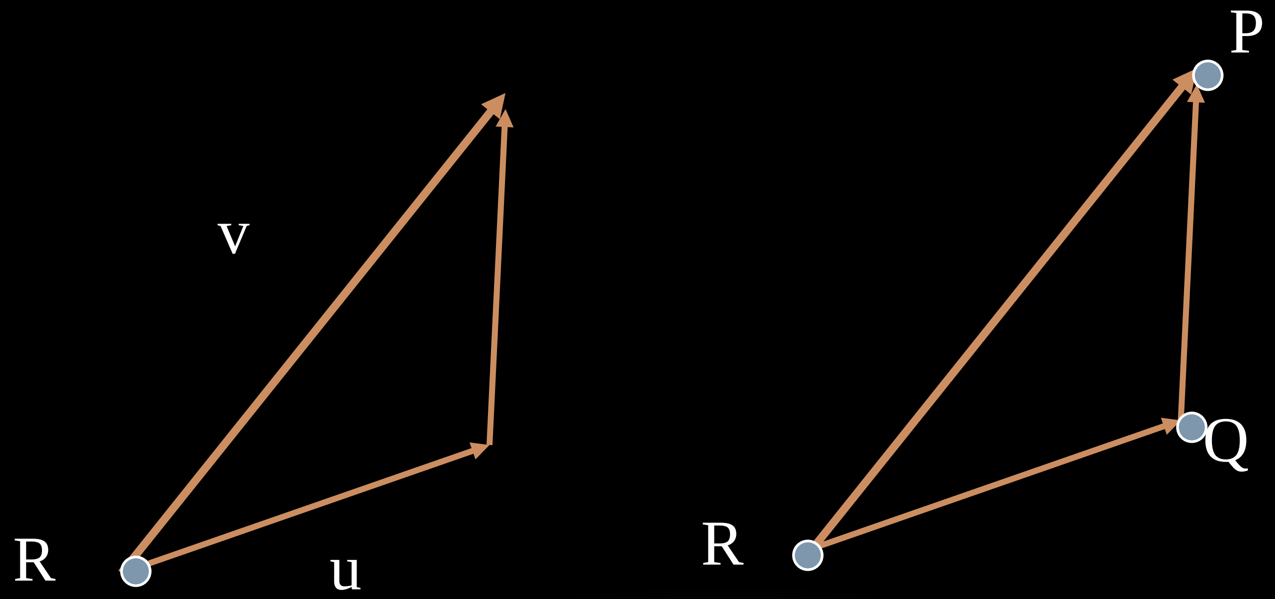
CONVEXITY

- An object is *convex* iff for any two points in the object all points on the line segment between these points are also in the object



PLANES

- A plane can be defined by a point and two vectors or by three points



VECTOR OPERATIONS DEFINED: ADDITION

- Vectors A and B, A = (u, v, w), B= (x, y, z)

A + B =

(u+x, v+y, w+z)

- Of course, subtraction is the same thing, just with the negation of the elements

VECTOR OPERATIONS DEFINED: NORMALIZATION

- Vectors A = (u, v, w)
- Normalization is just changing the length/magnitude to 1
- Normalized A =

  A / |A|, where |A| is the length of A

  e.g. A / sqrt($u^2 + v^2 + w^2$)

VECTOR OPERATIONS DEFINED: DOT PRODUCT

- Vectors A and B, A = (u, v, w), B= (x, y, z)

A • B =

(ux + vy + wz)

Result is a scalar

COOL USES OF THE DOT PRODUCT

Find the angle between two vectors:

$$\cos\theta = A \cdot B / |A| * |B|$$

If the vectors have been normalized in advance, then $|A| = |B| = 1$ and the equation reduces to:

$$\cos\theta = A \cdot B$$

So, $\theta = \arccos(A \cdot B)$

But, sometimes what we really want is $\cos\theta$, so we don't always need to take it further

MORE COOL USES OF THE DOT PRODUCT

Find a vector's magnitude: sqrt(A • A)

Two vectors are perpendicular if: A • B = 0

Two vectors are parallel if: A • B = |A| * |B|

Parallel but pointing in opposite directions: A • B = - |A| * |B|

Angle between vectors is between -90 and 90 degrees: A • B > 0

VECTOR OPERATIONS DEFINED: CROSS PRODUCT

- Vectors A and B, A = (u, v, w), B= (x, y, z)

A x B =

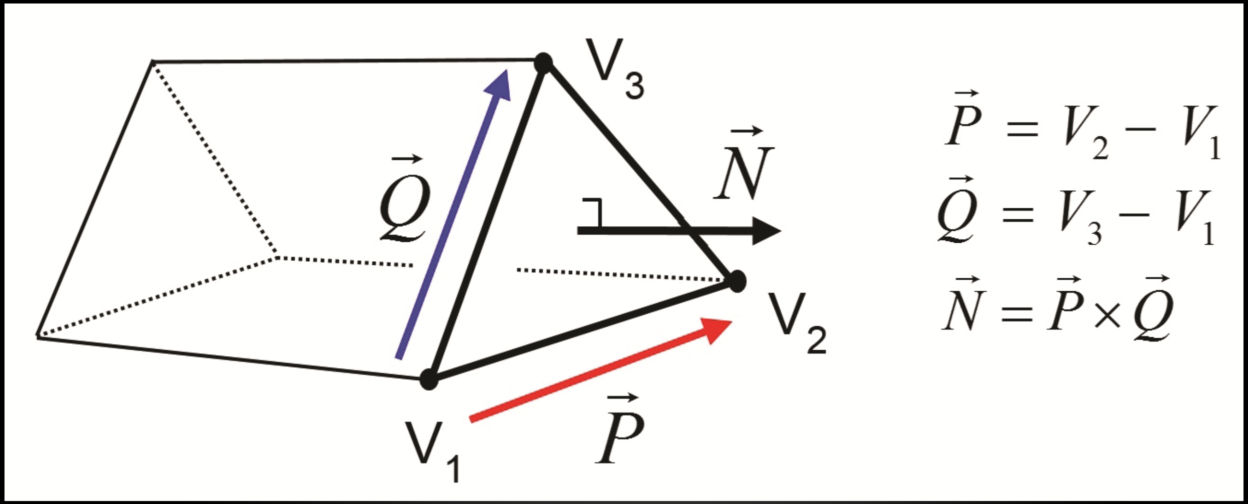
(vz-wy, wx-uz, uy-vx)

COOL USES OF THE CROSS PRODUCT

- Finding the normal vector to a plane

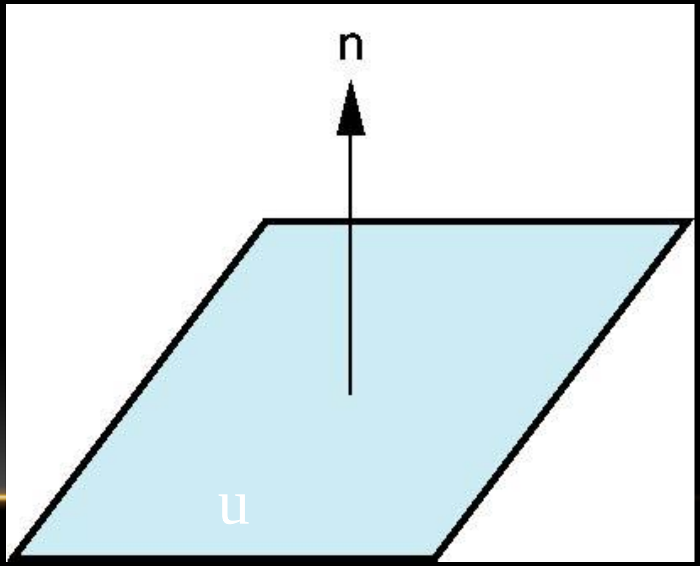
A x B =

(vz-wy, wx-uz, uy-vx)



NORMALS

- Every plane has a vector n normal (perpendicular, orthogonal) to it
- We will use these extensively when calculating lighting and other effects



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SUMMARY

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