Theory of Computation, CSCI 438 spring 2022
Variations of Turing Machines, pg. 176, March 24th

1. Consider a machine that is similar to a TM but has three possible moves: left, right or stay. Show that this type of Turing machine recognizes the class of Turing-recognizable languages.

Proof:

⇒ (only if) $\mathcal{L}(TM) \subseteq \mathcal{L}(TM_{Stay})$

Suppose that a language is Turing-recognizable. Then some Turing Machine recognizes the language. This TM can be considered a TM_{Stay} which simply doesn't use a stay move.

With careful thinking about it, you will see that this new TM_{Stay} will recognize the same language as the regular TM recognized. Thus when a language is Turing-recognizable, it is recognized by a TM_{Stay}.

⇐ (if) $\mathcal{L}(TM_{Stay}) \subseteq \mathcal{L}(TM)$

Suppose that a language is recognized by a TM_{Stay}. Consider the following TM which:

1. Marks the beginning of the tape with a $ (some symbol which wasn't in Γ of TM_{Stay}), shifting all non-blank symbols one position to the right.
2. When TM_{Stay} makes a right move, the regular TM makes the exact same move. When TM_{Stay} makes a left move, the regular TM makes the exact same move. When the regular TM_{Stay} makes a stay move, the TM does 2 moves, first a right and then a left. In this way the regular TM has simulated the stay move.

Clearly this TM will recognize the same language as the language which TM_{Stay} recognized. Thus when a language is recognized by a TM_{Stay} it is also Turing-recognizable.

2. Consider a machine that is similar to a TM but has an infinite tape in both directions. Say that the entire tape is blank, except for the input which is in contiguous cells with the read/write head pointed at the leftmost input symbol. Show that this type of Turing machine recognizes the class of Turing-recognizable languages. (Exercise 3.11)

A language is Turing-recognizable iff some Turing Machine with an infinite tape recognizes it.

Proof:
⇒ (only if)
Suppose that a language is Turing-recognizable. Since it is Turing-recognizable, some Turing Machine recognizes the language. Consider the following $TM_{infiniteBothWays}$ machine which:

1. Moves left and marks the position to the left of the input with a $ (some symbol not in Γ).
2. When the regular TM makes a right move, the $TM_{infiniteBothWays}$ does the same. When the regular TM makes a left move, the $TM_{infiniteBothWays}$ moves left.
   - If it reached the marked position, it moves right, acting like a regular TM which has moved left when it is already at the left end of the tape.
   - If it didn't reach the marked position, it does nothing more.

With some thinking about it, you will see that this new $TM_{infiniteBothWays}$ will recognize the same language as the regular TM recognized. Thus when a language is Turing-recognizable, it is recognized by a $TM_{infiniteBothWays}$.

⇐ (if)
Suppose that a language is recognized by a $TM_{infiniteBothWays}$. Consider the following regular TM which:

1. Marks the beginning of the tape with a $ (some symbol not in Γ), shifting all non-blank symbols one position to the right.
2. Returns to the position to the right of the $.
3. When the $TM_{infiniteBothWays}$ makes a right move, the regular TM makes the exact same move. When the $TM_{infiniteBothWays}$ makes a left move, the regular TM checks if it has encountered the leftmost end of the tape.
   - If it has, it shifts all non-blank symbols one position to the right and travels back to the beginning of the tape to simulate the left move.
   - If it hasn't, it just makes the exact same move as the $TM_{infiniteBothWays}$.

With some thinking about it, you will see that this new regular TM will recognize the same language as $TM_{infiniteBothWays}$ recognized. Thus when a language is recognized by a $TM_{infiniteBothWays}$, it is also Turing-recognizable.

3. Prove the following theorem.

Theorem: A language is Turing-recognizable iff some Turing Machine with reset (and no left move) recognizes it.

Proof:

$\Leftarrow$ (if) $\mathcal{L}(TM_{Reset}) \subseteq \mathcal{L}(TM)$

Suppose that a language is recognized by a TM_{Reset}. Consider the following TM which:

1. Marks the beginning of the tape with a \$ (some symbol which wasn't in Γ of TM_{Reset}), shifting all non-blank symbols one position to the right.
    2. When TM_{Reset} makes a right move, the regular TM makes the exact same move. When the regular TM_{Reset} makes a reset move, the TM travels left until it reaches the beginning of tape marker. It then moves right. In this way is has simulated the reset move.

Clearly this TM will recognize the same language as the language which TM_{Reset} recognized. Thus when a language is recognized by a TM_{Reset} it is also Turing-recognizable.

$\Rightarrow$ (only if) $\mathcal{L}(TM) \subseteq \mathcal{L}(TM_{Reset})$

Suppose that a language is Turing-recognizable. Then some Turing Machine recognizes the language. Consider the following TM_{Reset} which:

- When the regular TM makes a right move, TM_{Reset} makes the exact same move.
- When the regular TM makes a left move from the leftmost position (it is already at the start of the tape), the TM_{Reset} does nothing.
- When the regular TM makes a left move and it is not in the leftmost position, TM_{Reset} does the following:
    1. Mark the current symbol and reset. The marking scheme could be, for each symbol $x \in \Gamma$ of the TM, define a new symbol which is x with a dot over it.
    2. Insert a \$ (some symbol not in Γ of the TM) at the beginning of the tape, shifting all non-blank symbols one position to the right. (Recall that this can be done by repeatedly reading a symbol, remembering it, writing whatever was remembered previously, and moving right.) While shifting all non-blank symbols to the right, when a marked symbol is encountered, TM_{Reset} writes the remembered symbol (the previous symbol) with a mark on it and removes the mark from the previously marked symbol. The remaining non-blank symbols are shifted as before.
    3. Reset.
    4. Move right across \$ and then across other symbols until the marked symbol is read. The read/write head will now be positioned as if it made a left move. Unmark the symbol when performing the next move.

With careful thinking about it, you will see that this new TM_{Reset} will recognize the same language as the regular TM recognized. Thus when a language is Turing-recognizable, it is recognized by a TM_{Reset}.