Theory of Computation, CSCI 438 spring 2022
More examples proving that languages are not regular, Feb. 7

1.53 Let Σ={0, 1, +, =} and

ADD ={x=y+z | x, y, z are binary integers, and x is the sum of y and z}.

Show that ADD is not regular.

Proof:

Suppose, by way of contradiction, that ADD is regular. Then the pumping lemma must hold for ADD. Let p be the pumping length. Consider the string s="$1^{p+1}=1^p+1^1$". Note that $s \in$ADD and $|s| \geq p$. Therefore, the pumping lemma guarantees that s can be divided into three pieces s=xyz, with |y|>0, |xy|≤p, and where $s_i=xy^iz \in$ ADD for all i≥0.

For any breakdown s=xyz with |xy|≤p, y must lie entirely in the first set of 1's. Thus, for all possible breakdowns we have

$x=1^{|x|}$,
$y=1^{|y|}$, and
z= "$1^{p-|xy|+1}=1^p+1^1$".

Consider string s_0 = xz = $1^{p-|y|+1}=1^p+1^1$". In order for s_0 to be in ADD the arithmetic must work. However, since |y|>0 the arithmetic does not work, $s_0 \notin$Y and the pumping lemma does not hold for ADD. Thus, ADD much not have been regular.

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1.47 Let $\Sigma=\{1,\#\}$ and let

$Y=\{w \mid w=x_1\#x_2\#...\#x_k$ for $k \geq 0$, each $x_i \in 1^*$, and $x_i \neq x_j$ for $i \neq j\}$

Prove that Y is not regular.

Proof:

Suppose, by way of contradiction, that Y is regular. Then the pumping lemma must hold for Y. Let p be the pumping length. Consider the string $s=1^p\#1^{p-1}\#1^{p-2}\#...\#1^0$. Note that $s \in Y$ and $|s| \geq p$. Therefore, the pumping lemma guarantees that s can be divided into three pieces $s=xyz$, with $|y|>0$, $|xy| \leq p$, and where $s_i=xy^iz \in Y$ for all $i \geq 0$.

For any breakdown $s=xyz$ with $|xy| \leq p$, y must lie entirely in the first set of 1's. Thus, for all possible breakdowns we have

$x=1^{|x|}$,

$y=1^{|y|}$, and

$z=1^{p-|xy|}\#1^{p-1}\#1^{p-2}\#...\#1^0$.

Consider string $s_0 = xz = 1^{p-|y|}\#1^{p-1}\#1^{p-2}\#...\#1^0$. In order for s_0 to be in Y it must be true that for each x_i and x_j, $i \neq j \rightarrow x_i \neq x_j$. However, since $|y|>0$ there is some x_i, $i>0$, where $x_i=x_1$. This means $s_0 \notin Y$ and the pumping lemma does not hold for Y. Thus, Y must not have been regular.

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Here is an alternative way to show that Y is not regular.

Note that the language $1^*\#1^*$ is regular. Call this language R.

Since regular languages are closed under complementation and intersection, if Y is regular then complement(Y) ∩ R would also be regular. However complement(Y) ∩ R = $\{1^n\#1^n\}$ which we know is not regular. Therefore, Y could not be regular.

Determine whether or not the following language is regular. If it is regular, prove it giving a DFA or an NFA for it; if it is not regular, prove it is not using the pumping lemma.

$$L=\{wtw \mid w, t \in \{0,1\}^+\}$$

L is not regular.

Proof:
Suppose, by way of contradiction, that L is regular. Then the pumping lemma must hold for L. Let p be the pumping length. Consider the string $s=0^P1^P10^P1^P$. Clearly $|s| \geq p$ and $s \in L$ where $w=0^P1^P$ and $t=1$. Therefore, the pumping lemma guarantees that s can be divided into three pieces s=xyz, with |y|>0, |xy|≤p, and where $s_i=xy^iz \in L$ for all i≥0.

For any breakdown s=xyz with |xy|≤p, y must lie entirely in the first set of 0's. Thus, for all possible breakdowns we have

$x=0^{|x|}$,
$y=0^{|y|}$, and
$z=0^{p-|xy|}1^P10^P1^P$.

Consider string $s_2 = xyyz = 0^{p+|y|}1^P10^P1^P$. In order for s_2 to be in L it must be of the form $\{wtw \mid w, t \in \{0,1\}^+\}$. Since s_2 ends with a '1', w must end with a '1'. Therefore, the first w must be at least $0^{p+|y|}1$. However, this contains more 0's than are possible at the end of the string. Thus, s_2 is not in L.

Since the only possible decomposition of s was with y lying entirely in the 0's, yet, for this decomposition, s could not be pumped, the pumping lemma does not hold for L and L must not have been regular.
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Note that for this proof to work, s had to be pumped up. If s was pumped down, the string $s_0 = xy^0z = xz = 0^{P-|y|}1^P10^P1^P$ would be in the language.

It is surprising that this language is NOT regular. It looks very much like the language $\{wtw \mid w, t \in \{0,1\}^*\}$, which is regular, however, now w must have at least one element. You might think to just make the length of w be 1. Then the string needs to begin and end with the same symbol. However, if a string doesn't begin and end with the same symbol, it still might be in L. Possibly w=01.

Another way to prove that L is not regular is to intersect the language with $0^+1^+10^+1^+$ and it is fairly easy to show that the result is not regular.

Problem 1.54 (page 91)
Consider the language F= $\{a^i b^j c^k \mid i, j, k \geq 0$ and if $i=1$ then $j=k\}$.

a. Show that F is not regular.

Proof that L is not regular:
Suppose, by way of contradiction, that F is regular. Then the pumping lemma must hold for F. Let p be the pumping length. Consider the string $s=ab^Pc^P$. Clearly $s \in F$ and $|s| \geq p$. Therefore, the pumping lemma guarantees that s can be divided into three pieces s=xyz, with $|y|>0$, $|xy| \leq p$, and where $xy^iz \in F$ for all $i \geq 0$.

There are 3 possible breakdowns s=xyz with $|xy| \leq p$.

Case 1. x is empty and y=a.
Case 2. x is empty and $y=ab^n$.
Case 3. y lies somewhere in the bs.

Case 1. x is empty and y=a.
In this case, all s_i's are in F. Thus, the pumping lemma holds for L and I can not use this method to show that F is not regular. Bummer!

(Case 2 & Case 3 do not allow pumping. However, the pumping lemma only says that one decomposition can be pumped, not that all of them can.)

FAILED PROOF!

Alternate proof that F is not regular:
Consider the language R = {ab*c*}. Clearly R is regular, as it is described by the regular expression ab*c*.

Recall that regular languages are closed under intersection.
$F \cap R = \{ab^nc^n \mid n \geq 0\}$.

Suppose, by way of contradiction, that F is regular. Then
$F \cap R = \{ ab^nc^n \mid n \geq 0\}$ is regular. However, we know from earlier examples that $\{ ab^nc^n \mid n \geq 0\}$ is not regular. Thus, F must not have been regular.
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b. Show that F acts like a regular language in the pumping lemma. In other words, give a pumping length p and demonstrate that F satisfies the three conditions of the pumping lemma for this value of p.

Consider the pumping lemma applied to F. Let p be the pumping length. Consider the string $s= aab^pc^p$. Clearly $s \in F$ and $|s| \geq p$. The pumping lemma guarantees that s can be divided into three pieces s=xyz where $xy_iz \in F$ for all $i \geq 0$, $|y|>0$, and $|xy| \leq p$.

Consider the decomposition:

$x=aab^n$,
$y=b^m$, $m>0$,
$z=b^{p-(n+m)}c^p$

All $s_i = xy_iz \in F$, $i \geq 0$. Thus the pumping lemma holds for F.

c. Explain why parts a and b do not contradict the pumping lemma.

The pumping lemma can be used to show that a language is not regular. It cannot be used to show that a language is regular. Therefore, the fact that the pumping lemma holds for F does not prove that F is regular.