Theory of Computation, CSCI 438 spring 2022
More examples proving that languages are not regular, Feb. 4

1. Prove that $L=\{ww^R \mid w\in\{a,b\}^*\}$ is not regular.
   (Note that the w^R is string w reversed.)

   Proof:
   Suppose, by way of contradiction, that L is regular. Then the pumping lemma must hold for L. Let p be the pumping length. Consider the string $s=a^pbba^p$. Clearly $s\in L$ and $|s| \geq p$. The pumping lemma guarantees that s can be divided into three pieces s=xyz where $xy^iz \in L$ for all $i\geq 0$, $|y|>0$ and $|xy|\leq p$.

   Consider all possible ways to divide s so that s=xyz with $|y|>0$ and $|xy|\leq p$.

   The only way is $x=a^{|x|}$, $y=a^{|y|}$ and $z=a^{|p-x-y|}\ bba^p$.(In other words, y must occur entirely in the a's.)

   Consider $xz= a^{|x|}\ a^{|p-x-y|}\ b^p =a^{p-|y|}bba^p$. By the pumping lemma $xz\in L$. However, there are fewer a's at the beginning of the string than at the end of the string, so xz can't be in L.

   There is no way to divide s into xyz so the pumping lemma holds. This means that the pumping lemma does not hold for L. Since the pumping lemma must hold for all regular languages, L must not be regular.

2. Exercise 1.29 b (page 88)

Use the pumping lemma to show that the following language is not regular.

$$A_2 = \{www \mid w \in \{a, b\}^*\}$$

Prove that $A_2 = \{www \mid w \in \{a, b\}^*\}$ is not regular.

Proof: Suppose, by way of contradiction, that A_2 is regular. Then the pumping lemma must hold for A_2. Let p be the pumping length. Consider the string $s=a^pba^pba^pb$. Clearly $s \in A_2$ with each $w=a^pb$. Also, $|s| \geq p$. The pumping lemma guarantees that s can be divided into three pieces s=xyz, with $|y|>0$, $|xy| \leq p$ and where $xy^iz \in A_2$ for all $i \geq 0$.

For any breakdown s=xyz with $|xy| \leq p$, y must lie entirely in the first set of a's. Thus we have $x=a^{|x|}$ where $|x| \geq 0$, $y = a^{|y|}$ where $|y|>0$, and $z=a^{p-|xy|}ba^pba^pb$.

Consider string $s_0 = xy^0z = xz$. This string will be $a^{|x|}a^{p-|xy|}ba^pba^pb = a^{p-|y|}ba^pba^pb$. This string cannot be A_2 because there are exactly three b's, so each w must consist of a set of a's followed by a b. However the first set of a's followed by a b has fewer a's than the second and third set of a's followed by a b. Thus $s_0 \notin A_2$. This means that the pumping lemma does not hold for A_2, so A_2 must not have been regular.

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Exercise 1.30 Describe the error in the following "proof" that 0*1* is not a regular language. (An error must exist because 0*1* *is* regular.) The proof is by contradiction. Assume that 0*1* is regular. Let p be the pumping length for 0*1* given by the pumping lemma. Choose s to be the string 0^p1^p. You know that s is a member of 0*1*, but Example 1.73, page 80, shows that s cannot be pumped. Thus you have a contradiction. So 0*1* is not regular.

EXAMPLE 1.73

Let B be the language $\{0^n1^n | n \geq 0\}$. We use the pumping lemma to prove that B is not regular. The proof is by contradiction.

Assume to the contrary that B is regular. Let p be the pumping length given by the pumping lemma. Choose s to be the string 0^p1^p. Because s is a member of B and s has length more than p, the pumping lemma guarantees that s can be split into three pieces, $s = xyz$, where for any $i \geq 0$ the string xy^iz is in B. We consider three cases to show that this result is impossible.

1. The string y consists only of 0s. In this case, the string $xyyz$ has more 0s than 1s and so is not a member of B, violating condition 1 of the pumping lemma. This case is a contradiction.
2. The string y consists only of 1s. This case also gives a contradiction.
3. The string y consists of both 0s and 1s. In this case, the string $xyyz$ may have the same number of 0s and 1s, but they will be out of order with some 1s before 0s. Hence it is not a member of B, which is a contradiction.

Thus a contradiction is unavoidable if we make the assumption that B is regular, so B is not regular. Note that we can simplify this argument by applying condition 3 of the pumping lemma to eliminate cases 2 and 3.

In this example, finding the string s was easy because any string in B of length p or more would work. In the next two examples, some choices for s do not work so additional care is required.

Example 1.73 shows that the language $\{0^n1^n | n \geq 0\}$ is not regular. In that language the string s= 0^p1^p could not be pumped. The string s can be pumped in the language 0*1*. If pumped down, there will be fewer 0's than 1's. If pumped up, there will be more 0's than 1's. In both cases, the new string is still in the language.