Theory of Computation, CSCI 438 spring 2022
Nonregular Languages, pg. 77-82, Feb. 2

Prove that the following two languages are not regular, and Problem 1.36 (page 89)

1. Prove that $L = \{a^n b^n : n \geq 0\}$ is not regular.

   Proof:
   Suppose, by way of contradiction, that L is regular. Then the pumping lemma must hold for L. Let p be the pumping length. Consider the string $s=a^p b^p$. Clearly $s \in L$ and $|s| \geq p$. The pumping lemma guarantees that s can be divided into three pieces s=xyz where $xy^i z \in L$ for all $i \geq 0$, $|y|>0$ and $|xy| \leq p$.

   Consider all possible ways to divide s so that s=xyz with $|y|>0$ and $|xy| \leq p$.

   The only way is $x=a^{|x|}$, $y=a^{|y|}$ and $z=a^{|p-x-y|} b^p$. (In other words, y must occur entirely in the a's.)

   Consider $xz= a^{|x|} a^{|p-x-y|} b^p = a^{p-|y|} b^p$. By the pumping lemma $xz \in L$. However, xz has fewer a's than b's so it can't be in L.

   There is no way to divide s into xyz so the pumping lemma holds. This means that the pumping lemma does not hold for L. Since the pumping lemma must hold for all regular languages, L must not be regular.

2. Prove that $L = \{a^n b^{2n} : n \geq 0\}$ is not regular.

Proof:
Suppose, by way of contradiction, that L is regular. Then the pumping lemma must hold for L. Let p be the pumping length. Consider the string $s=a^p b^{2p}$. Clearly $s \in L$ and $|s| \geq p$. The pumping lemma guarantees that s can be divided into three pieces s=xyz where $xy^i z \in L$ for all $i \geq 0$, $|y|>0$ and $|xy| \leq p$.

Consider all possible ways to divide s so that s=xyz with $|y|>0$ and $|xy| \leq p$.

The only way is $x=a^{|x|}$, $y=a^{|y|}$ and $z=a^{|p-x-y|} b^{2p}$. (Once again, y must occur entirely in the a's.)

Consider $xy^{p+1}z= a^{|x|} a^{|y|} a^{|y|} \ldots a^{|y|} a^{|y|} a^{|p-x-y|} b^{2p}$ where $a^{|y|}$ is repeated p+1 times. By the pumping lemma $xy^{p+1}z \in L$. However, $xy^{p+1}z = a^{2p}b^{2p}$ which does not have twice as many b's as a's, so it can't be in L.

There is no way to divide s into xyz so the pumping lemma holds. This means that the pumping lemma does not hold for L. Since the pumping lemma must hold for all regular languages, L must not be regular.

Problem 1.36 (page 89)
Let $B_n = \{a^k \mid k$ is a multiple of $n\}$. Show that for each $n \geq 1$, the language B_n is regular.

A language is regular iff there is a DFA which recognizes it.

However, in class we have shown that:

DFAs and NFAs recognize the same class of languages

Regular expressions describe, and NFAs recognize, the same class of languages

Thus, I can use any of these formalisms to show that B_n is regular.

$B_1 = \{\varepsilon, a, aa, aaa, aaaa, \ldots\}$ which can be described by the regular expression: a*
$B_2 = \{\varepsilon, aa, aaaa, aaaaaa, \ldots\}$ which can be described by the regular expression: (aa)*
$B_3 = \{\varepsilon, aaa, aaaaaa, aaaaaaaaa, \ldots\}$
which can be described by the regular expression: (aaa)*

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B_n can be described by the regular expression: (aa…a)* where there are n a's.

Alternative demonstration that B_n is regular:

B_n is recognized by the following NFA: