Theory of Computation, CSCI 438 spring 2022
Relation between NFAs and DFAs, pg. 54-63, Jan. 24

We want to know the relation between NFAs and DFAs in terms of the languages which they can describe.

Two questions:

Lang. described by DFA $\subseteq$ Lang. described by NFA?
Lang. described by NFA $\subseteq$ Lang. described by DFA?

Algorithm to go from an NFA to a DFA that accepts the same language

1. The start state is all the states that can be accessed from the start state of the NFA, following ε–transitions.
2. Repeat until all edges are defined:
   Take vertex $\{q_i, q_j, \ldots q_k\}$and alphabet symbol *a*
   I. Find $\delta^*(q_i,a)$, $\delta^*(q_j,a)$, … $\delta^*(q_k,a)$.
      (δ^* because ε-transitions can be applied before or after the transition)
   II. Union the results.
   III. If there is no state like the result of the union, create a new one.
   IV. Draw an edge from state $\{q_i, q_j, \ldots q_k\}$ to the new or other state. Label this *a*.
3. Make any state which contains one of the original final states, final.

Theorem: A language is regular if and only if (iff) some NFA recognizes it. (This is a corollary in the text. Theorem 1.39, page 55, proves the hard direction.)

"only if" ($\Rightarrow$) A language is regular only if some NFA recognizes it.

Proof: Let L be a regular language. Since L is regular, there is some DFA, M, which recognizes L (that is, $\mathcal{L}(M) = L$). Any DFA can be seen as an NFA where all transitions go to exactly one state and there are no ε-transitions. Thus, there is an NFA which recognizes the language.

"if" ($\Leftarrow$) A language is regular if some NFA recognizes it.

Proof: Assume that a language is recognized by the NFA $M = (Q, \Sigma, \delta, q_0, F)$.

Define a function E, which takes a state and returns a set of states:

$E(r) = \{q \mid q \text{ can be reached from } r \text{ by traveling along 0 or more } \varepsilon \text{ arrows}\}$

Define the DFA $M' = (\mathcal{P}(Q), \Sigma, \delta', E(q_0), \{R \in \mathcal{P}(Q) \mid \exists\, q \in R \wedge q \in F\})$
where δ' is defined as follows:

$\delta'(R, a) = \{q \in Q \mid q \in E(\delta(x, a))\ \exists\, x \in E(r) \text{ for some } r \in R\}$

(I'm using capital R for a state in the DFA, because each state in the DFA is a set of states of the NFA. A capital suggests a set of states, while small letters suggest a single state.)

Clearly M' recognizes the same language that was recognized by the NFA. Thus there is a DFA that recognizes the language, and the language is regular.