**Theory of Computation, CSCI 438 spring 2022**
**Properties of regular languages, pg. 44-47, Jan. 19**

Languages are sets of strings on an alphabet. Therefore we can define the following operations on languages:

- Union, $\cup$
- Intersection, $\cap$
- Complementation, bar over the set
- Difference (Set difference) , \
- Concatenation, $^{\circ}$
- Star (Kleene closure) , *

Closure under an operation: In mathematics, a set is said to be closed under some operation if the operation on members of the set produces a member of the set.

Question: If L is regular is L (L complement) regular? (In other words, is the class of regular languages closed under complementation?)

Proof: Let L be a regular language. Since L is regular there is a DFA M which accepts L. Let M= $(Q,\Sigma, \delta, q_0, F)$.

Consider the DFA M'= $(Q,\Sigma, \delta, q_0, Q\backslash F)$. Claim: $\mathcal{L}$ (M') = $\overline{L}$

Let w be a string on the alphabet $\Sigma$. That is, $w \in \Sigma^*$. There are two possibilities:

- $w \in L$
- $w \notin L$

For $w \in L$, it must be true that starting in $q_0$ and consuming w using the transition function $\delta$ puts the machine M into a state in F. That is, $\delta^*( q_0,w) \in F$. This means that $\delta^*( q_0,w) \notin Q\backslash F$. Therefore, $w \notin \mathcal{L}$ (M').

For $w \notin L$, it must be true that starting in $q_0$ and consuming w using the transition function $\delta$ puts the machine M into a state not in F. That is, $\delta^*( q_0,w) \notin F$. This means $\delta^*( q_0,w) \in Q\backslash F$. Therefore, $w \in \mathcal{L}$ (M').

Since $w \in L$ implies $w \notin \mathcal{L}$ (M') and $w \notin L$ implies $w \in \mathcal{L}$ (M'), $\mathcal{L}$ (M') = $\overline{L}$.

Since M' is a DFA that recognizes the complement of L, the complement of a regular language L is also regular. Therefore, the set of regular languages is closed under complementation.

Question: Say that $L_1$ and $L_2$ are defined on the same alphabet. If $L_1$ and $L_2$ are regular is $L_1$ intersect $L_2$ regular? (In other words, is the class of regular languages closed under intersection?)

The set of regular languages is closed under intersection. The proof for this is very similar to the proof that regular languages are closed under union.

Proof:
Say $L_1$ and $L_2$ are regular languages. Since $L_1$ and $L_2$ are regular, there must be DFAs $M_1=(Q,\Sigma,\delta_q,q_0,F_q)$ and $M_2=(P,\Sigma,\delta_p,p_0,F_p)$ that recognize $L_1$ and $L_2$, respectively. (That is, $\mathcal{L}(M_1)=L_1$ and $\mathcal{L}(M_2)=L_2$.)

Consider a new DFA, $M_{intersection}$, defined as follows:

$M_{intersection}$, $= (QxP, \Sigma, \delta`, (q_0, p_0), F_{intersection})$

where $\delta`$ is defined by:

$$\delta`((q_i, p_j),a) = (\delta q(q_i,a), \delta p(p_j,a))$$

and F $M_{intersection}$ is defined by

$$F_{intersection}, = \{(q_i, p_j)\} \ q_i \in F_q \text{ and } p_j \in F_p\}$$

With some thought, it can be see that $M_{intersection}$, accepts $L_1 \cap L_2$. Thus $L_1 \cap L_2$ is regular and regular languages are closed under intersection.