Theory of Computation, CSCI 438 spring 2022
No new readings, March 4

1. Fill out the two columns, telling if the language is regular/context-free, and how you would prove this.

	Language	The language is regular	The language is context free
1	$L=\{a^n b^n \mid n \geq 0\}$	No, pumping lemma for regular languages	Yes, create a grammar
2	$L=\{a^i b^j \mid i<j\}$	No, pumping lemma for regular languages	Yes, create a grammar
3	$L=\{a^i b^j \mid i \geq 0 \text{ and } j>0\}$	Yes, create a dfa/nfa	Yes, all regular languages are also context free
4	$L=\{a^n b^n c^n \mid n \geq 0\}$	No, pumping lemma for regular languages	No, pumping lemma for context-free languages
5	Language of all strings xyz where x, y and z are elements of {a,b}* and $z = x^r$	Tricky, language is regular since x and y can be empty. Create a dfa/nfa	Yes, all regular languages are also context free
6	Language consisting of all strings on {0,1}* with more 0's than 1's.	No, pumping lemma for regular languages	Yes, create a pda or grammar
7	$L=\{ww^rww^r\}$	No, pumping lemma for regular languages	No, pumping lemma for context-free languages ($L=\{ww^rvv^r\}$ would be context free)
8	Language consisting of all strings of 0's and 1's where any 0 is followed by a 1.	Yes, create a dfa/nfa	Yes, all regular languages are also context free
9	$L=\{a^n b^j : n = j^2\}$	No, pumping lemma for regular languages	No, pumping lemma for context-free languages
10	Language consisting of all strings of 0's and 1's not containing the string 010	Yes, create a dfa/nfa	Yes, all regular languages are also context free

2. Use the pumping lemma to prove that the following is not context free.

$L = \{w : n_a(w) \le n_b(w) \le n_c(w)\}$.

Claim L is not context-free.

Proof: Suppose, by way of contradiction, that L is context-free. Then the pumping lemma must hold for L. Let p be the pumping length. Consider the string $s= a^p b^p c^p$. Clearly $s \in L$ and $|s| \ge p$. The pumping lemma guarantees that s can be divided into five pieces $s=uvxyz$ where $|vxy| \le p$, $|vy|>0$ and $s_i=uv^ixy^iz \in L$ for all $i \ge 0$.

Consider all possible breakdowns $s=uvxyz$ where $|vxy| \le p$, $|vy|>0$.

Case 1: vxy is entirely in one set of symbols.
Case 2: vxy includes both a's and b'a, or both b'a and c's.

Consider each case separately.

Case 1: vxy is entirely in one set of symbols.
If vxy is entirely in the a's or entirely in the b's, pump up to get a string which has more a's or more b's than c's, so is not in L.

If vxy is entirely in the c's, pump down to get a string which has fewer c's than b's, so is not in the language.

Case 2: vxy includes both a's and b'a, or both b'a and c's.
If vxy includes both a's and b's, pump up and there will not be enough c's so the string won't be in the language.

If vxy includes both b's and c's, pump down and there will be too many a's so the string won't be in the language.

It has been shown that there is no way to divide s into uvxyz so the pumping lemma holds. Thus, the pumping lemma does not hold for L, and L must not have been context-free.

3. Consider $L=\{a^n\#a^m \mid m \geq n^2\}$. Create a push down automaton for L or use the pumping lemma for context-free languages to prove that L is not context free so there is no push down automaton for it.

Answer: L is not context free. General plan: Since we have the ≥ when the vy is in the first set of a's, pump up, but when it is in the second set of a's, pump down – or vice versa. Four cases are necessary.

Claim L is not context-free.

Proof: Suppose, by way of contradiction, that L is context-free. Then the pumping lemma must hold for L. Let p be the pumping length. Consider the string s= "a^p#ap^2". Clearly $s \in L$ and $|s| \geq p$. The pumping lemma guarantees that s can be divided into five pieces $s=uvxyz$ where $|vxy| \leq p$, $|vy|>0$ and $s_i=uv^ixy^iz \in L$ for all $i \geq 0$.

Consider all possible breakdowns $s=uvxyz$ where $|vxy| \leq p$, $|vy|>0$.

Case 1: vxy is entirely in the first set of a's.
Case 2: vxy is entirely in the final set of a's.
Case 3: vxy includes #. This case can be broken into two cases,
 Case 3a: v or y includes #
 Case 3b: v is in the first set of a's and y is in the final set of as.

Consider each case separately.

Case 1: vxy is entirely in the first set of a's
In this case consider $s_2=uv^2xy^2z = a^{p+|vy|}\#a^{p\wedge 2}$. Notice that $(p+|vy|)^2 > p^2$, so $s_s \notin L$.
(Alternatively, consider $s_p=uv^pxy^pz = a^{p+|vy|*p}\#a^{p\wedge 2}$. Notice that $(p+|vy|*p)^2 > p^2$, so $s_p \notin L$.)

Case 2: vxy is entirely in the final set of a's.
In this case consider $s_0=uv^0xy^0z = a^p\#a^{p\wedge 2-|vy|}$. Notice that $p^2-|vy|<p^2$, so $s_0 \notin L$.

Case 3a: v or y contains a #.
In this case $s_2=uv^2xy^2z$ will have more than one # so cannot be in L.

Case 3b v is in the first set of a's and y is in the final set of as.
In this case consider $s_2=uv^2xy^2z = a^{p+|v|}\#a^{p\wedge 2+|y|}$. For each added 'a' which occurs before the #, at least p^2 'a's must be added after the # sign. This can't occur because $|vxy| \leq p$. Thus s_2 can not be in L.

It has been shown that there is no way to divide s into uvxyz so the pumping lemma holds. Thus, the pumping lemma does not hold for L, and L must not have been context-free.