Theory of Computation, CSCI 438 spring 2022
Pumping lemma for context-free languages, pages 123- 127, March 2

1. Prove that L = $\{a^n b^n c^n \mid n \geq 0\}$ is not context-free. (This is example 2.36, page 126, in the text.)

Proof: Suppose, by way of contradiction, that L is context-free. Then the pumping lemma must hold for L. Let p be the pumping length. Consider the string
s= $a^p b^p c^p$. Clearly $s \in L$ and $|s| \geq p$. The pumping lemma guarantees that s can be divided into five pieces s=uvxyz where $|vxy| \leq p$, $|vy|>0$ and $s_i = uv^i xy^i z \in L$ for all $i \geq 0$. Consider all possible breakdowns s=uvxyz where $|vxy| \leq p$, $|vy|>0$.

Case 1: vxy occurs entirely in the a's, b's or the c's.
Case 2: vxy occurs within the a's and b's.
Case 3: vxy occurs within the b's and c's.

Consider each case separately.

Case 1: vxy occurs entirely in the a's, b's or the c's.
Say that vxy occurs entirely in the a's. Then s_0 will contain fewer of a's than b's or c's. Thus s_0 is not in L.
The case where vxy lies entirely in the b's or c's is handled similarly.

Case 2: vxy occurs within the a's and b's.
Say that vxy occurs within the a's and b's. Then s_0 will contain n c's, but either fewer than n a's and fewer than n b's. Thus s_0 is not in L.

Case 3: vxy occurs within the b's and c's.
This case is handled similar to case 2.

It has been shown that there is no way to divide s into uvxyz so the pumping lemma holds. Thus, the pumping lemma does not hold for L, and L must not have been context-free.

2. Use the pumping lemma to prove that the following is not context free.

$L = \{a^n b^m c^k : m>n,\ n>k \text{ and } m,n,k \geq 0\}$.

Claim L is not context-free.

Proof: Suppose, by way of contradiction, that L is context-free. Then the pumping lemma must hold for L. Let p be the pumping length. Consider the string $s= a^p\ b^{p+1}\ c^{p-1}$. Clearly $s \in L$ and $|s| \geq p$. The pumping lemma guarantees that s can be divided into five pieces $s=uvxyz$ where $|vxy| \leq p$, $|vy|>0$ and $s_i=uv^ixy^iz \in L$ for all $i \geq 0$.

Consider all possible breakdowns $s=uvxyz$ where $|vxy| \leq p$, $|vy|>0$.

Case 1: vxy lies entirely in the a's.
Case 2: vxy includes both a's and b's.
Case 3: vxy lies entirely in the b's.
Case 4: vxy includes both b's and c's.
Case5: vxy lies entirely in c's.

Consider each case separately.

Case 1: vxy lies entirely in the a's.
If vxy lies entirely in the a's, pump up to get a string which has the same or more a's than b's. That is, $s_2= a^{p+|vy|}b^{p+1}\ c^{p+2}$. Since $|vy|>0$, there are too many a's, and $s_2 \notin L$.

Case 2: vxy includes both a's and b's.

Case 2a. v or y includes both a's and b's.

If v or y include both a's and b's, pump up, to get a's and b's interleaved. That is, if $v=a^rb^s$, then $s_2= a^pb^sa^rb^{p+1+s}c^{p-1}$, which is not in L. Similarly, if y includes both a's and b', $y= a^rb^s$, $s_2= a^pb^sa^rb^{p+1+s}c^{p-1}$ which isn't in L.

Case 2b, Neither v, nor y, includes both a's and b's.

In this case v must lie entirely in the a's and y must lie in the b's. If $|v|>0$, pump down so there are no longer fewer c's than a's ($s_0= a^{p-|v|}b^wc^{p-1}$). If $|v|=0$, then $|y|>0$, pump down, reducing the number of b's, so there are no longer more b's than a's ($s_0= a^pb^{p+1-|y|}c^{p-1}$).

Case 3: vxy lies entirely in the b's.
If vxy lies entirely in the b's, pump down to get a string which has the same or more a's than b's. That is, $s_0= a^pb^{p+1-|vy|}c^{p+2}$. Since $|vy|>0$, there are too many b's, and $s_0 \notin L$.

Case 4: vxy includes both b's and c's.

Case 4a. v or y includes both b's and c's.

Similar to Case 2a, if v or y include both b's and c's, pumping up will cause the b's and c's to be interleaved, so $s_2 \notin L$.

Case 4b, Neither v, nor y, includes both b's and c's.

In this case v must lie entirely in the b's and y must lie in the c's. If $|v|>0$, pump down so there are no more b's than a's ($s_0= a^p b^{p+1-|v|} c^w \notin L$). If $|v|=0$, then $|y|>0$, pump up increasing the number of c's, so there are no longer fewer c's than a's ($s_2= a^p b w^{|} c^{p-1+|v|} \notin L$).

Case 5: vxy lies entirely in the c's.
If vxy lies entirely in the c's, pump up to get a string which has the same or more c's than a's. That is, $s_2= a^p b^{p+1|} c^{p-1+|vy|} \notin L$.

Thus there are no decompositions s=uvxyz that allow the string s to be pumped. Thus, the pumping lemma does not hold for L, and L must not have been context-free.