SECTION 1.6 Introduction to Proofs

This introduction applies jointly to this section and the next (1.7).

Learning to construct good mathematical proofs takes years. There is no algorithm for constructing the proof of a true proposition (there is actually a deep theorem in mathematical logic that says this). Instead, the construction of a valid proof is an art, honed after much practice. There are two problems for the beginning student—figuring out the key ideas in a problem (what is it that really makes the proposition true?) and writing down the proof in acceptable mathematical language.

Here are some general things to keep in mind in constructing proofs. First, of course, you need to find out exactly what is going on — why the proposition is true. This can take anywhere from ten seconds (for a really simple proposition) to a lifetime (some mathematicians have spent their entire careers trying to prove certain conjectures). For a typical student at this level, tackling a typical problem, the median might be somewhere around 15 minutes. This time should be spent looking at examples, making tentative assumptions, breaking the problem down into cases, perhaps looking at analogous but simpler problems, and in general bringing all of your mathematical intuition and training to bear.

It is often easiest to give a proof by contradiction, since you get to assume the most (all the hypotheses as well as the negation of the conclusion), and all you have to do is to derive a contradiction. Another thing to try early in attacking a problem is to separate the proposition into several cases: proof by cases is a valid technique, if you make sure to include all the possibilities. In proving propositions, all the rules of inference are at your disposal, as well as axioms and previously proved results. Ask yourself what definitions, axioms, or other theorems might be relevant to the problem at hand. The importance of constantly returning to the definitions cannot be overstated!

Once you think you see what is involved, you need to write down the proof. In doing so, pay attention both to content (does each statement follow logically? are you making any fallacious arguments? are you leaving out any cases or using hidden assumptions?) and to style. There are certain conventions in mathematical proofs, and you need to follow them. For example, you must use complete sentences and say exactly what you mean. (An equation is a complete sentence, with "equals" as the verb; however, a good proof will usually have more English words than mathematical symbols in it.) The point of a proof is to convince the reader that your line of argument is sound, and that therefore the proposition under discussion is true; put yourself in the reader's shoes, and ask yourself whether you are convinced.

Most of the proofs called for in this exercise set are not extremely difficult. Nevertheless, expect to have a fairly rough time constructing proofs that look like those presented in this solutions manual, the textbook, or other mathematics textbooks. The more proofs you write, utilizing the different methods discussed in this section, the better you will become at it. As a bonus, your ability to construct and respond to nonmathematical arguments (politics, religion, or whatever) will be enhanced. Good luck!

1. We must show that whenever we have two odd integers, their sum is even. Suppose that a and b are two odd integers. Then there exist integers s and t such that $a = 2s + 1$ and $b = 2t + 1$. Adding, we obtain $a + b = (2s + 1) + (2t + 1) = 2(s + t + 1)$. Since this represents $a + b$ as 2 times the integer $s + t + 1$, we conclude that $a + b$ is even, as desired.

3. We need to prove the following assertion for an arbitrary integer n: "If n is even, then n^2 is even." Suppose that n is even. Then $n = 2k$ for some integer k. Therefore $n^2 = (2k)^2 = 4k^2 = 2(2k^2)$. Since we have written n^2 as 2 times an integer, we conclude that n^2 is even.

5. We can give a direct proof. Suppose that $m + n$ is even. Then $m + n = 2s$ for some integer s. Suppose that $n + p$ is even. Then $n + p = 2t$ for some integer t. If we add these [this step is inspired by the fact that we want to look at $m + p$], we get $m + p + 2n = 2s + 2t$. Subtracting $2n$ from both sides and factoring, we have

27. We can set this up in two-column format.

Step	**Reason**
1. $\forall x(P(x) \wedge R(x))$	Premise
2. $P(a) \wedge R(a)$	Universal instantiation using (1)
3. $P(a)$	Simplification using (2)
4. $\forall x(P(x) \rightarrow (Q(x) \wedge S(x)))$	Premise
5. $Q(a) \wedge S(a)$	Universal modus ponens using (3) and (4)
6. $S(a)$	Simplification using (5)
7. $R(a)$	Simplification using (2)
8. $R(a) \wedge S(a)$	Conjunction using (7) and (6)
9. $\forall x(R(x) \wedge S(x))$	Universal generalization using (5)

28. We can set this up in two-column format. The proof is rather long but straightforward if we go one step at a time.

Step	**Reason**
1. $\exists x \neg P(x)$	Premise
2. $\neg P(c)$	Existential instantiation using (1)
3. $\forall x(P(x) \vee Q(x))$	Premise
4. $P(c) \vee Q(c)$	Universal instantiation using (3)
5. $Q(c)$	Disjunctive syllogism using (4) and (2)
6. $\forall x(\neg Q(x) \vee S(x))$	Premise
7. $\neg Q(c) \vee S(c)$	Universal instantiation using (6)
8. $S(c)$	Disjunctive syllogism using (5) and (7), since $\neg\neg Q(c) \equiv Q(c)$
9. $\forall x(R(x) \rightarrow \neg S(x))$	Premise
10. $R(c) \rightarrow \neg S(c)$	Universal instantiation using (9)
11. $\neg R(c)$	Modus tollens using (8) and (10), since $\neg\neg S(c) \equiv S(c)$
12. $\exists x \neg R(x)$	Existential generalization using (11)

29. Let p be "It is raining"; let q be "Yvette has her umbrella"; let r be "Yvette gets wet." Then our assumptions are $\neg p \vee q$, $\neg q \vee \neg r$, and $p \vee \neg r$. Using resolution on the first two assumptions gives us $\neg p \vee \neg r$. Using resolution on this and the third assumption gives us $\neg r$, so Yvette does not get wet.

30. Assume that this proposition is satisfiable. Using resolution on the first two clauses allows us to conclude [illegible] in other words, we know that q has to be true. Using resolution on the last two clauses allows us to conclude $\neg q \vee \neg q$; in other words, we know that $\neg q$ has to be true. This is a contradiction. So this proposition is not satisfiable.

31. The argument is valid. We argue by contradiction. Assume that Superman does exist. Then he is not omnipotent, and he is not malevolent (this follows from the fourth sentence). Therefore by (the contrapositives of) the two parts of the second sentence, we conclude that he is able to prevent evil, and he is willing to prevent evil. By the first sentence, we therefore know that Superman does prevent evil. This contradicts the fifth sentence. Since we have arrived at a contradiction, our original assumption must have been false, so we conclude finally that Superman does not exist.