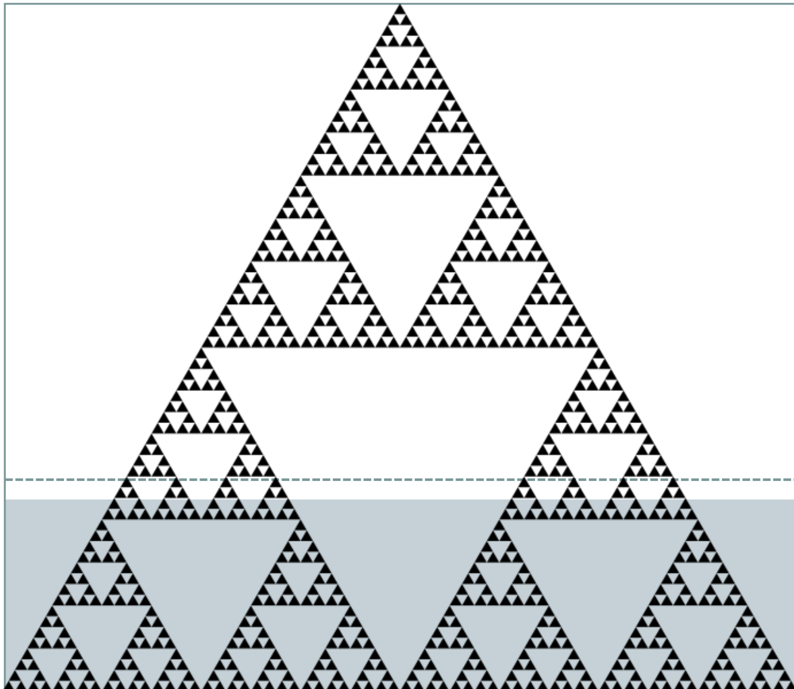
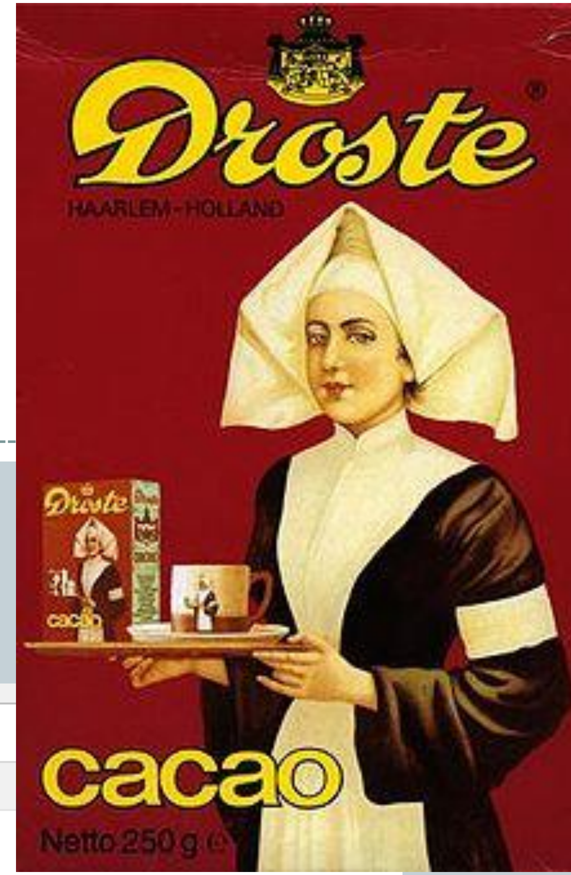
Recursion

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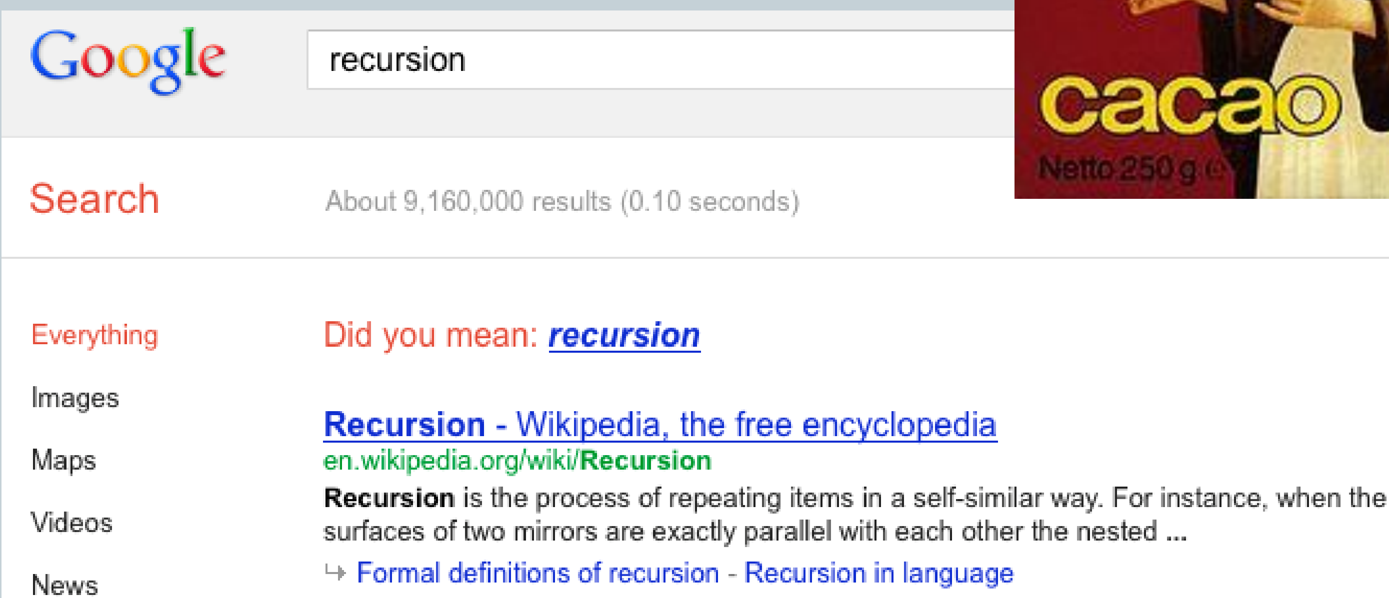
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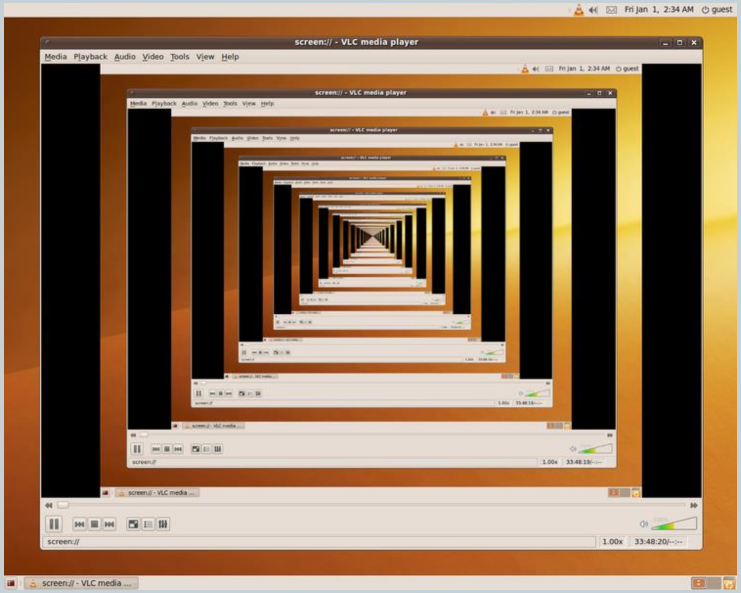
Did you mean: recursion
Recursion - Wikipedia, the free encyclopedia
en.wikipedia.org/wiki/Recursion
Recursion is the process of repeating items in a self-similar way. For instance, when the surfaces of two mirrors are exactly parallel with each other the nested ...
Formal definitions of recursion - Recursion in language

Outline

- Recursion
  - A method calling itself
  - All good recursion must come to an end
  - A powerful tool in computer science
    - Allows writing elegant and easy to understand algorithms
  - A new way of thinking about a problem
    - Divide and conquer
  - A powerful programming paradigm
  - Related to mathematical induction
- Example applications
  - Factorial
  - Binary search
  - Pretty graphics
  - Sorting things



Mathematical Induction

- Prove a statement involving an integer N
  - Base case: Prove it for small N (usually 0 or 1)
  - Induction step:
    - Assume true for size N-1
    - Prove it is true for size N

- Example:
  - Prove T(N) = 1 + 2 + 3 + … + N = N(N + 1) / 2 for all N
  - Base case: T(1) = 1 = 1(1 + 1) / 2
  - Induction step:
    - Assume true for size N – 1: 1 + 2 + … + N-1 = T(N - 1) = (N - 1)(N) / 2
    - T(N) = 1 + 2 + 3 + … + N-1 + N

= (N – 1)(N) / 2 + N

= (N – 1)(N) / 2 + 2N / 2

= (N – 1 + 2)(N) / 2

= (N + 1)(N) / 2

Hello Recursion

- Goal: Compute factorial N! = 1 * 2 * 3 ... * N
  - Base case: 0! = 1
  - Induction step:
    - Assume we know (N – 1)!
    - Use (N – 1)! to find N!

```
4! = 4 * 3 * 2 * 1 = 24

4! = 4 * 3!
3! = 3 * 2!
2! = 2 * 1!
1! = 1 * 0!
0! = 1
```

```
import sys

def fact(N):
    if N == 0:          <- base case
        return 1
    return fact(N - 1) * N          <- induction step

if __name__ == "__main__":
    N = int(sys.argv[1])
    print(str(N) + "! = " + str(fact(N)))
```

Instrumented Factorial

```
def fact(N):
    print("start, fact " + str(N))
    if N == 0:
        print("end base, fact " + str(N))
        return 1
    step = fact(N - 1)
    print("end, fact " + str(N))
    return step * N
```

```
start, fact 4
start, fact 3
start, fact 2
start, fact 1
start, fact 0
end base, fact 0
end, fact 1
end, fact 2
end, fact 3
end, fact 4
4! = 24
```

5 levels of fact()

Recursion vs. Iteration

- Recursive algorithms also have an iterative version

```
def fact(N):
   if N == 0:
      return 1
   return fact(N - 1) * N
```

Recursive algorithm

```
def fact(N):
   result = 1
   for i in range(1, N+1):
      result *= i
   return result
```

Iterative algorithm

- Reasons to use recursion:
  - Code is more compact and easier to understand
  - Easer to reason about correctness
- Reasons not to use recursion:
  - If you end up recalculating things repeatedly (stay tuned)
  - Processor with very little memory (e.g. 8051 = 128 bytes)

A Useful Example of Recursion

- Binary search
  - Given an array of N sorted numbers
  - Find out if some number *t* is in the list
  - Do it faster than going linearly through the list, i.e. O(N)
- Basic idea:
  - Like playing higher/lower number guessing:

Me	You
I'm thinking of a number between 1 and 100.	50
Higher	75
Lower	63
Higher	69
Higher	72
You got it	Wow I'm super smart!

Binary Search

- Binary search algorithm
  - Find midpoint of sorted array
  - Is that element the one you're looking for?
    - If yes, you found it!
  - If target is < midpoint, search lower half
  - If target is > midpoint, search upper half
- Example: Is 5 in this sorted array?

1	2	2	5	8	9	14	14	50	88	89

```
target (value)    = 5
low (index)       = 0
high (index)      = 10
midpoint (index) = (0 + 10) / 2 = 5
```

Binary Search

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  - Find midpoint of sorted array
  - Is that element the one you're looking for?
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  - If target is < midpoint, search lower half
  - If target is > midpoint, search upper half
- Example: Is 5 in the sorted array?

1	2	2	5	8	9	14	14	50	88	89

```
target (value)    = 5
low (index)       = 0
high (index)      = 4
midpoint (index) = (0 + 4) / 2 = 2
```

Binary Search

- Binary search algorithm
  - Find midpoint of sorted array
  - Is that element the one you're looking for?
    - If yes, you found it!
  - If target is < midpoint, search lower half
  - If target is > midpoint, search upper half
- Example: Is 5 in the sorted array?

1	2	2	5	8	9	14	14	50	88	89

```
target (value)    = 5
low (index)       = 3
high (index)      = 4
midpoint (index) = (3 + 4) / 2 = 3
```

Binary Search

- Binary search algorithm
  - Find midpoint of sorted array
  - Is that element the one you're looking for?
    - If yes, you found it!
  - If target is < midpoint, search lower half
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- Example: Is 5 in the sorted array?

1	2	2	5	8	9	14	14	50	88	89

```
YES.  Element at new midpoint is target!
```

Binary Search, Recursive Algorithm

```
def binarySearch(target, low, high, d):
   mid = int((low + high) / 2)
   print("low", low, "high", high, "mid", mid)

   if d[mid] == target:
      return True

   if high < low:
      return False

   if d[mid] < target:
      return binarySearch(target, mid + 1, high, d)
   else:
      return binarySearch(target, low, mid - 1, d)

if __name__ == "__main__":
   d   = [1, 2, 2, 5, 8, 9, 14, 14, 50, 88, 89]
   target = int(sys.argv[1])
   print("found " + str(target) + "? " + str(binarySearch(target, 0, len(d)-1, d)))
```

Things to Avoid

- Missing base case

```
def fact(N):
    return fact(N - 1) * N
```

- No convergence guarantee

```
def badIdea(N):
   if N == 1):
      return 1.0
   return badIdea(1 + N/2) + 1.0/N
```

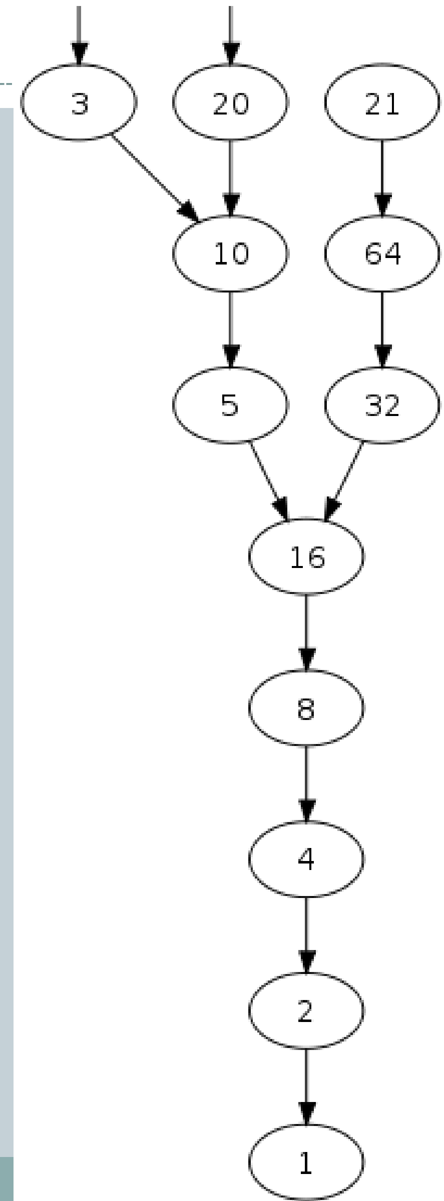
```
% python FactBad.py 5
Traceback (most recent call last):
  File "FactBad.py", line 20, in <module>
    print(str(N) + "! = " + str(fact(N)))
  File "FactBad.py", line 15, in fact
    return fact(N - 1) * N
  File "FactBad.py", line 15, in fact
    return fact(N - 1) * N
  File "FactBad.py", line 15, in fact
    return fact(N - 1) * N
  [Previous line repeated 996 more times]
RecursionError: maximum recursion depth
exceeded
```

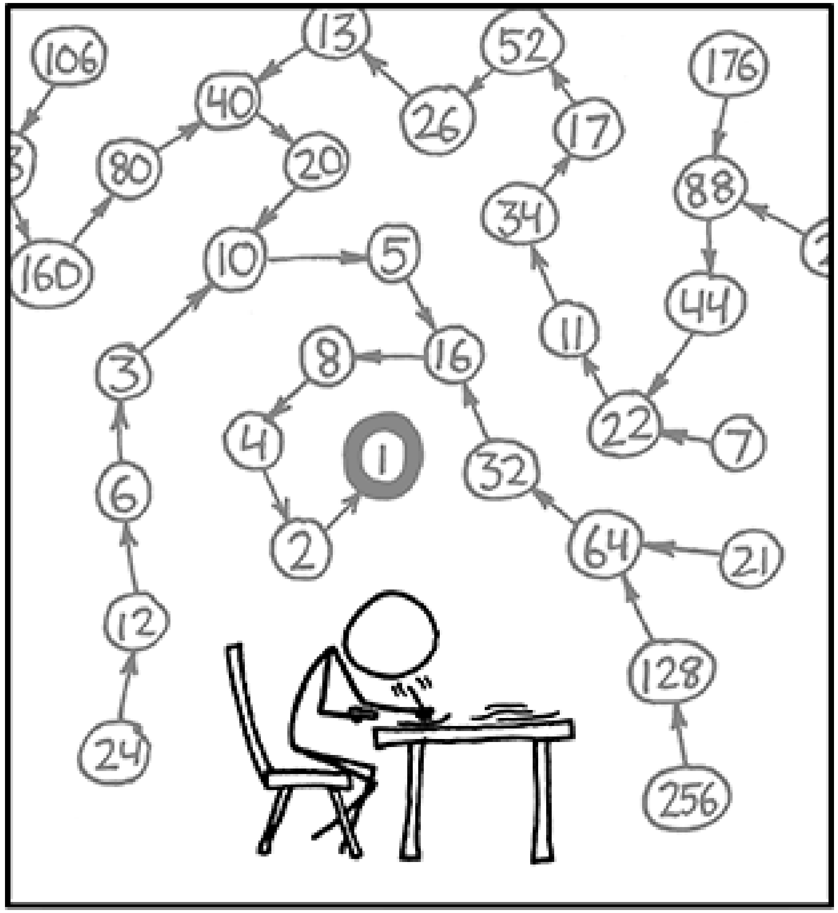
- Both result in infinite recursion = stack overflow

Sometimes We Don't Know...

- Collatz sequence
  - If N = 1, stop
  - If N is even, divide by 2
  - If N is odd, multiply by 3 and add 1
  - e.g. 24 12 6 3 10 5 16 8 4 2 1
  - No one knows if this terminates for all N!

```
def collatz(N):
   print(N)
   if N == 1:
      return
   elif N % 2 == 0:
      collatz(int(N / 2))
   else:
      collatz(3 * N + 1)
```

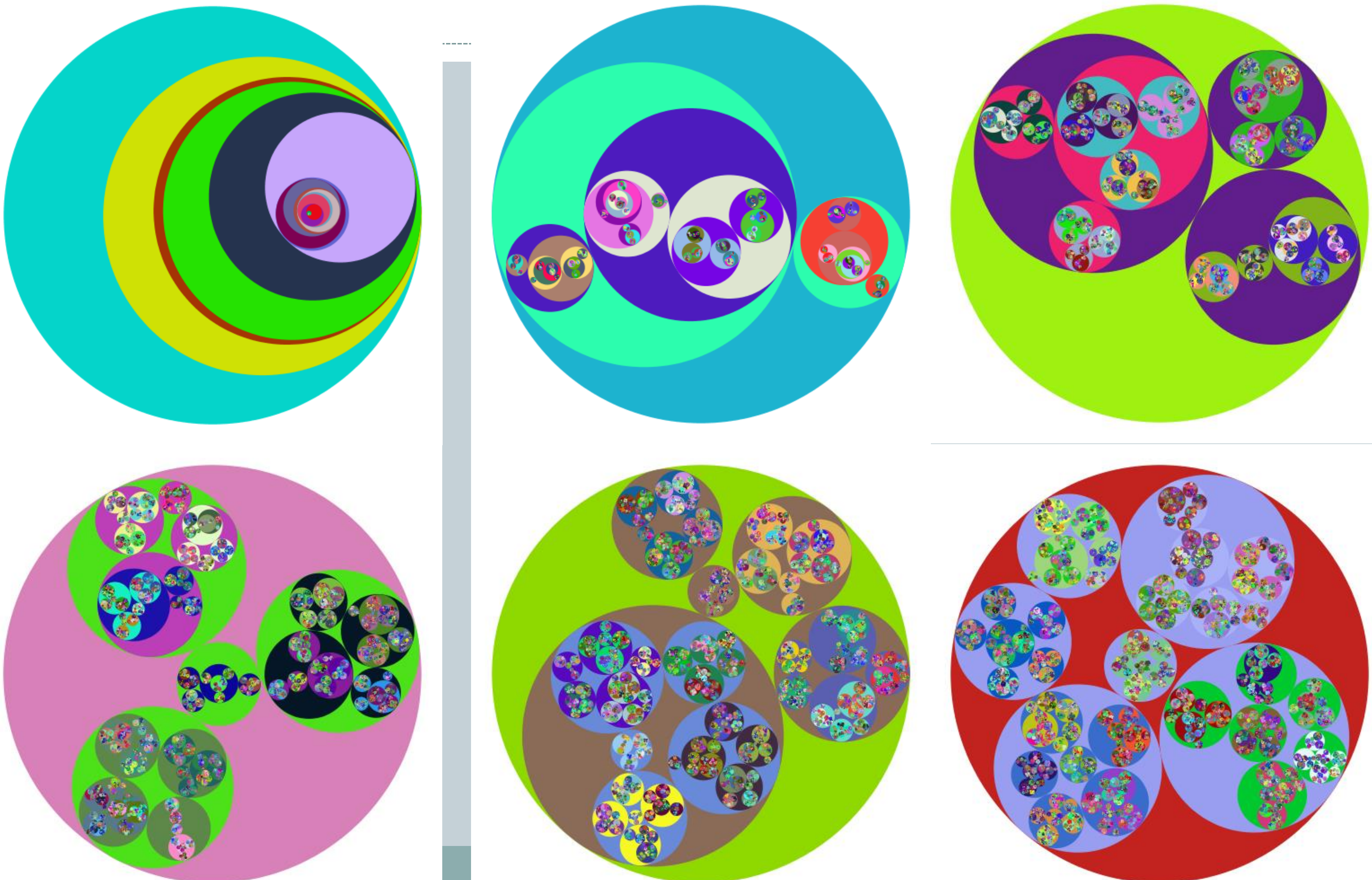




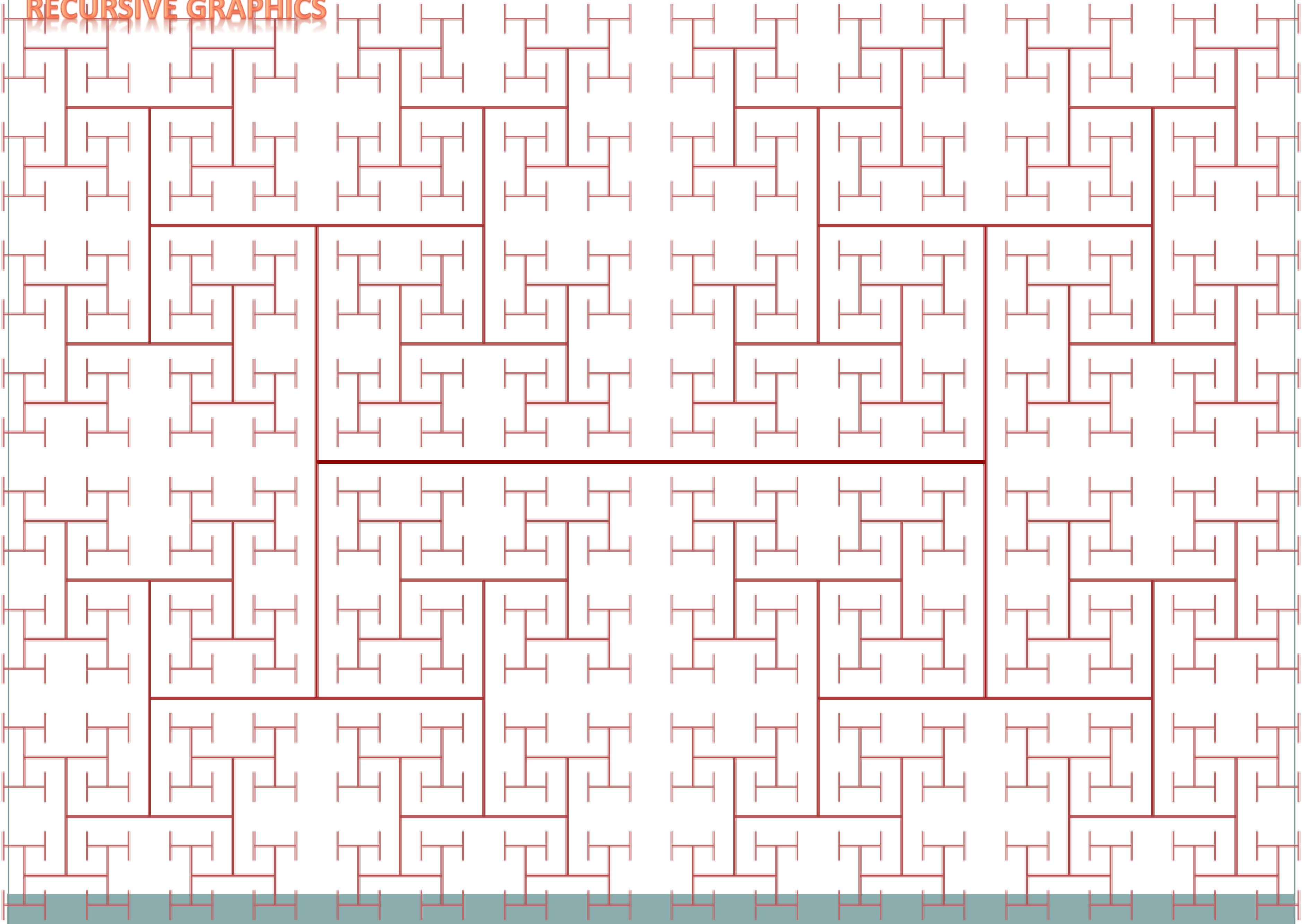
THE COLLATZ CONJECTURE STATES THAT IF YOU PICK A NUMBER, AND IF IT'S EVEN DIVIDE IT BY TWO AND IF IT'S ODD MULTIPLY IT BY THREE AND ADD ONE, AND YOU REPEAT THIS PROCEDURE LONG ENOUGH, EVENTUALLY YOUR FRIENDS WILL STOP CALLING TO SEE IF YOU WANT TO HANG OUT.

http://xkcd.com/710/

Recursive Graphics

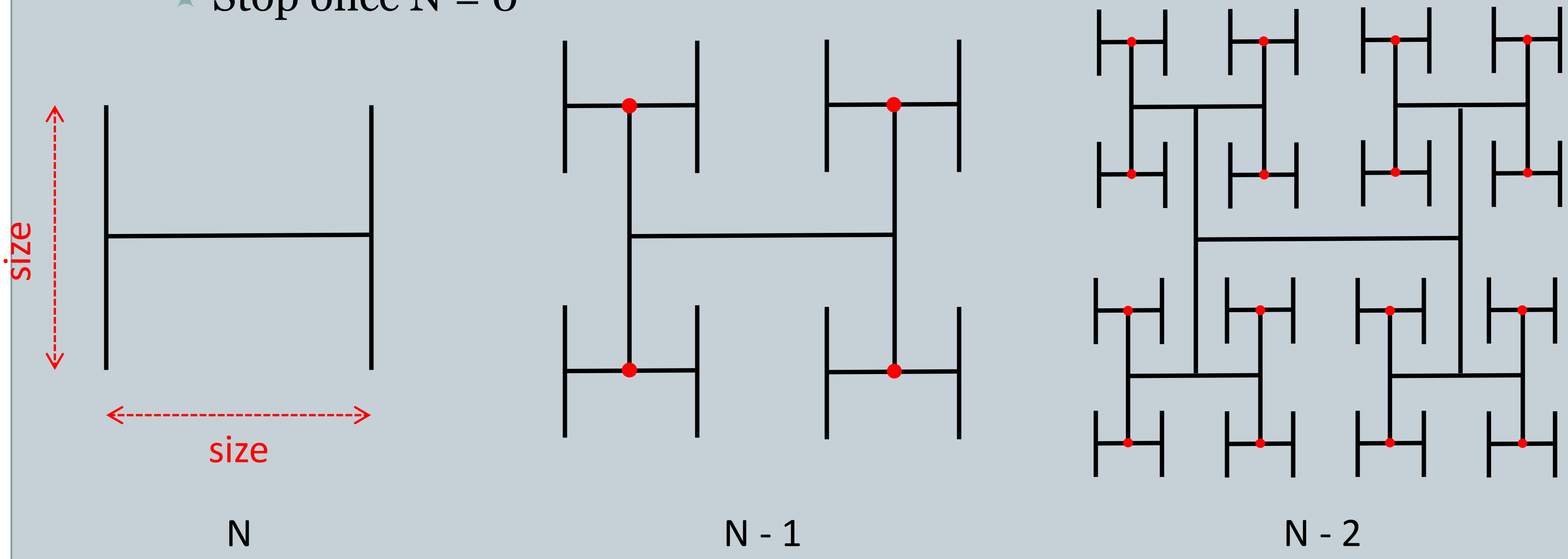


RECURSIVE GRAPHICS



H-tree

- H-tree of order N
  - Draw an H
  - Recursively draw 4 H-trees
    - One at each "tip" of the original H, half the size
    - Stop once N = 0



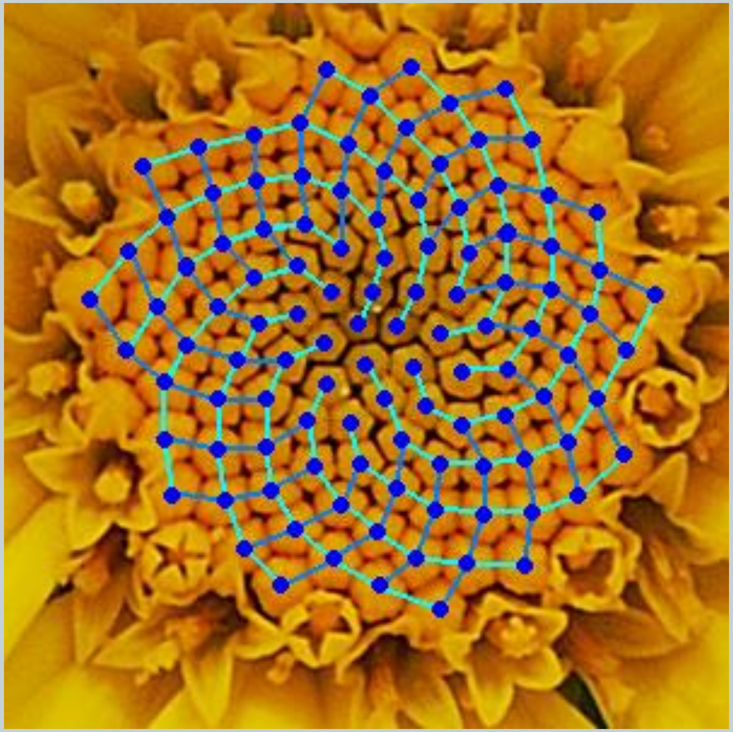
Fibonacci Numbers

- 0, 1, 1, 2, 3, 5, 8, 13, 21, 34, 55, 89, 144, 233, ...

$F_0 = 0$
$F_1 = 1$
$F_n = F_{n-1} + F_{n-2}$

Fibonacci numbers.
A natural fit for recursion?

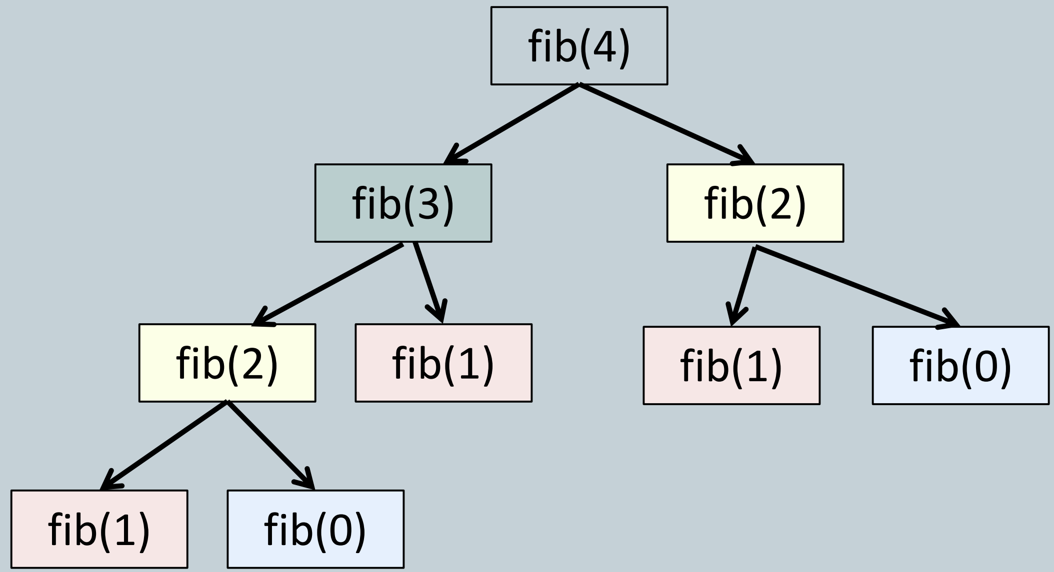
```
def fib(n):
   if n == 0:
      return 0
   if n == 1:
      return 1
   return fib(n - 1) + fib(n -2)
```



Yellow Chamomile head showing the arrangement in 21 (blue) and 13 (aqua) spirals.

Trouble in Recursion City…

N	time, fib(N)
10	0.000
20	0.002
30	0.011
40	0.661
41	1.080
42	1.748
43	2.814
44	4.531
45	7.371
46	11.860
47	19.295
48	31.319
49	50.668
50	81.542



Bad news bears:

Order of growth = Exponential!

Efficient Fibonacci Version

0, 1, 1, 2, 3, 5, 8, 13, 21, 34, 55, 89, 144, 233, 377

- Remember last two numbers
  - Use F_{n-2} and F_{n-1} to get F_n

Efficient Fibonacci Version

0, 1, 1, 2, 3, 5, 8, 13, 21, 34, 55, 89, 144, 233, 377

- Remember last two numbers
  - Use F_{n-2} and F_{n-1} to get F_n

Efficient Fibonacci Version

0, 1, 1, 2, 3, 5, 8, 13, 21, 34, 55, 89, 144, 233, 377

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Efficient Fibonacci Version

- Remember last two numbers
  - Use F_{n-2} and F_{n-1} to get F_n

0, 1, 1, 2, 3, 5, 8, 13, 21, 34, 55, 89, 144, 233, 377

```
def fibFast(n):
   n2 = 0
   n1 = 1
   if n == 0:
      return 0
   for i in range(1, n):
      n0 = n1 + n2
      n2 = n1
      n1 = n0
   return n1
```

N	time, fib(N)
50	0.001
100	0.001
200	0.001
400	0.001
10,000,000	0.010
20,000,000	0.016
40,000,000	0.028
80,000,000	0.051
160,000,000	0.096

Summary

- Recursion
  - A method calling itself
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  - A powerful tool in computer science
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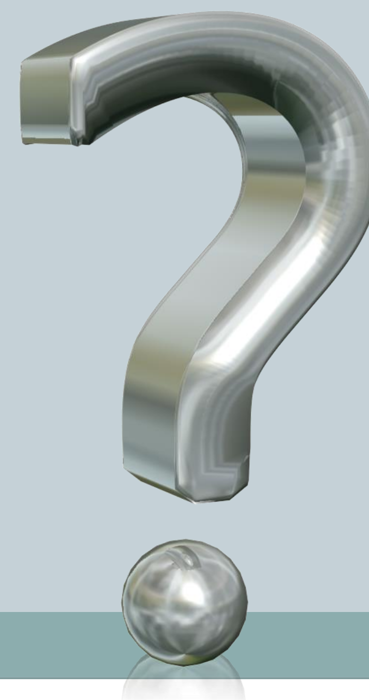

Your Turn

Here is a recursive definition for exponentiation. Write a recursive method to implement this definition. The test main is provided for you:

Fast Exponentiation

Recursive Definition:

$$a^n = \begin{cases} 1, & \text{if } n = 0, \\ (a^{\lfloor n/2 \rfloor})^2 & \text{if } n > 0 \text{ and } n \text{ is even}, \\ (a^{\lfloor n/2 \rfloor})^2 a & \text{if } n \text{ is odd}. \end{cases}$$

```
import sys

def fastExp(a, n):
    # Your code goes here…

if __name__ == "__main__":
    a = int(sys.argv[1])
    n = int(sys.argv[2]);
    print(a, " raised to the ", n, " is: ", fastExp(a, n))
```

- Open Moodle, go to CSCI 136, Section 11
- Open the dropbox for today – Activity 4: Recursion
- Drag and drop your program file to the Moodle dropbox
- You get: 1 point if you turn in something, 2 points if you turn in something that is correct.