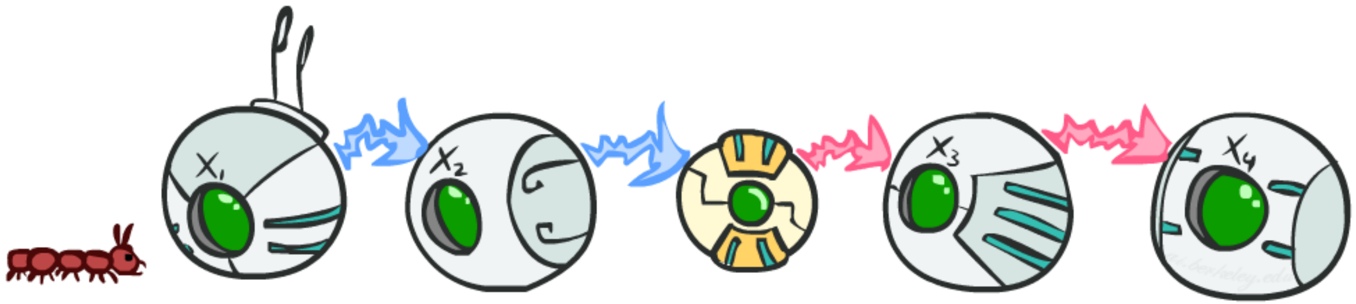
# CSCI 446: Artificial Intelligence

## Markov Models

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[These slides were created by Dan Klein and Pieter Abbeel for CS188 Intro to AI at UC Berkeley.  All CS188 materials are available at http://ai.berkeley.edu.]

# Today

- Probability Revisited
  - Independence
  - Conditional Independence
- Markov Models

# Independence

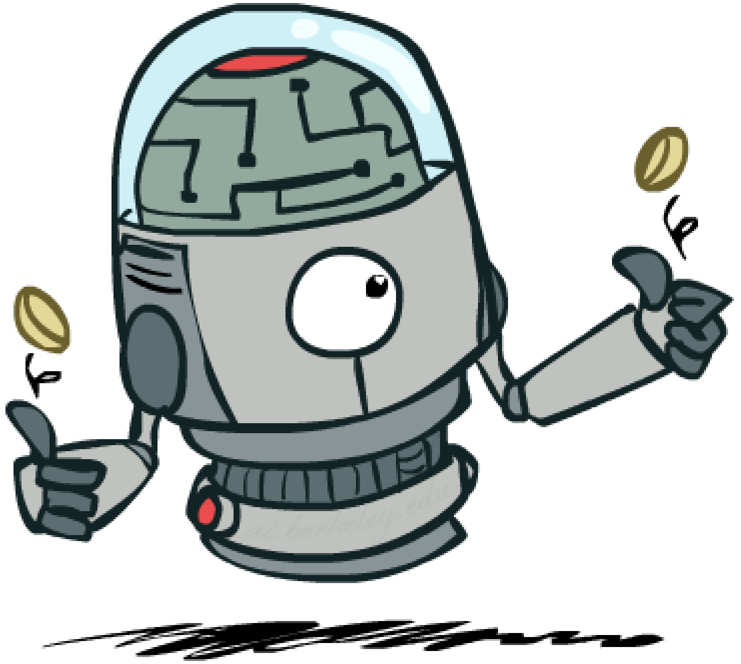
- Two variables are *independent* in a joint distribution if:

$$P(X,Y) = P(X)P(Y)$$

$$X \perp\!\!\!\perp Y$$

$$\forall x, y\, P(x,y) = P(x)P(y)$$

  - Says the joint distribution *factors* into a product of two simple ones
  - Usually variables aren't independent!



- Can use independence as a *modeling assumption*
  - Independence can be a simplifying assumption
  - *Empirical* joint distributions: at best "close" to independent
  - What could we assume for {Weather, Traffic, Cavity}?

- Independence is like something from CSPs: what?

# Example: Independence?

$$P(T)$$

| T | P |
|---|---|
| hot | 0.5 |
| cold | 0.5 |

$$P_1(T, W)$$

| T | W | P |
|---|---|---|
| hot | sun | 0.4 |
| hot | rain | 0.1 |
| cold | sun | 0.2 |
| cold | rain | 0.3 |

$$P_2(T, W) = P(T)P(W)$$

| T | W | P |
|---|---|---|
| hot | sun | 0.3 |
| hot | rain | 0.2 |
| cold | sun | 0.3 |
| cold | rain | 0.2 |

$$P(W)$$

| W | P |
|---|---|
| sun | 0.6 |
| rain | 0.4 |

# Example: Independence

- N fair, independent coin flips:

$P(X_1)$

| | |
|---|---|
| H | 0.5 |
| T | 0.5 |

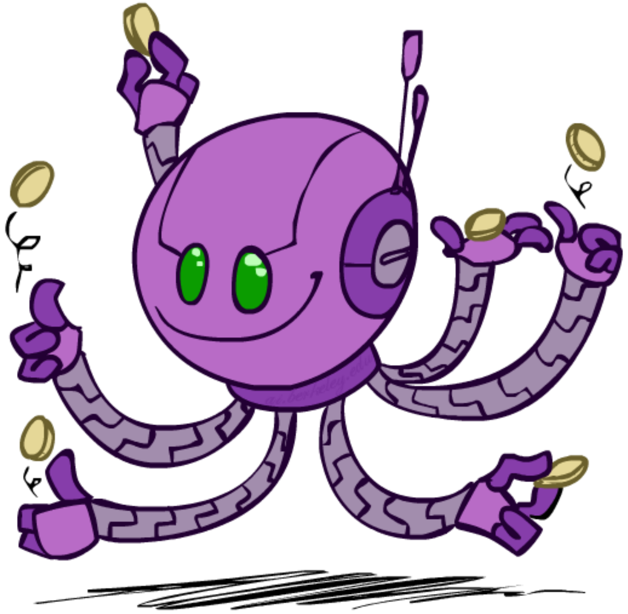
$P(X_2)$

| | |
|---|---|
| H | 0.5 |
| T | 0.5 |

$\cdots$

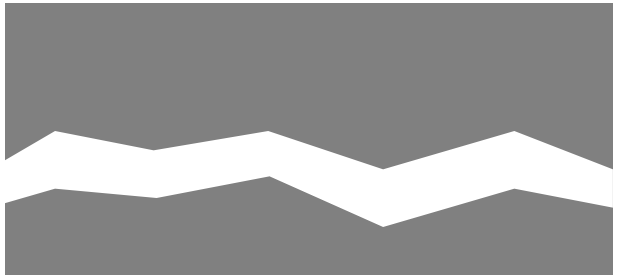
$P(X_n)$

| | |
|---|---|
| H | 0.5 |
| T | 0.5 |

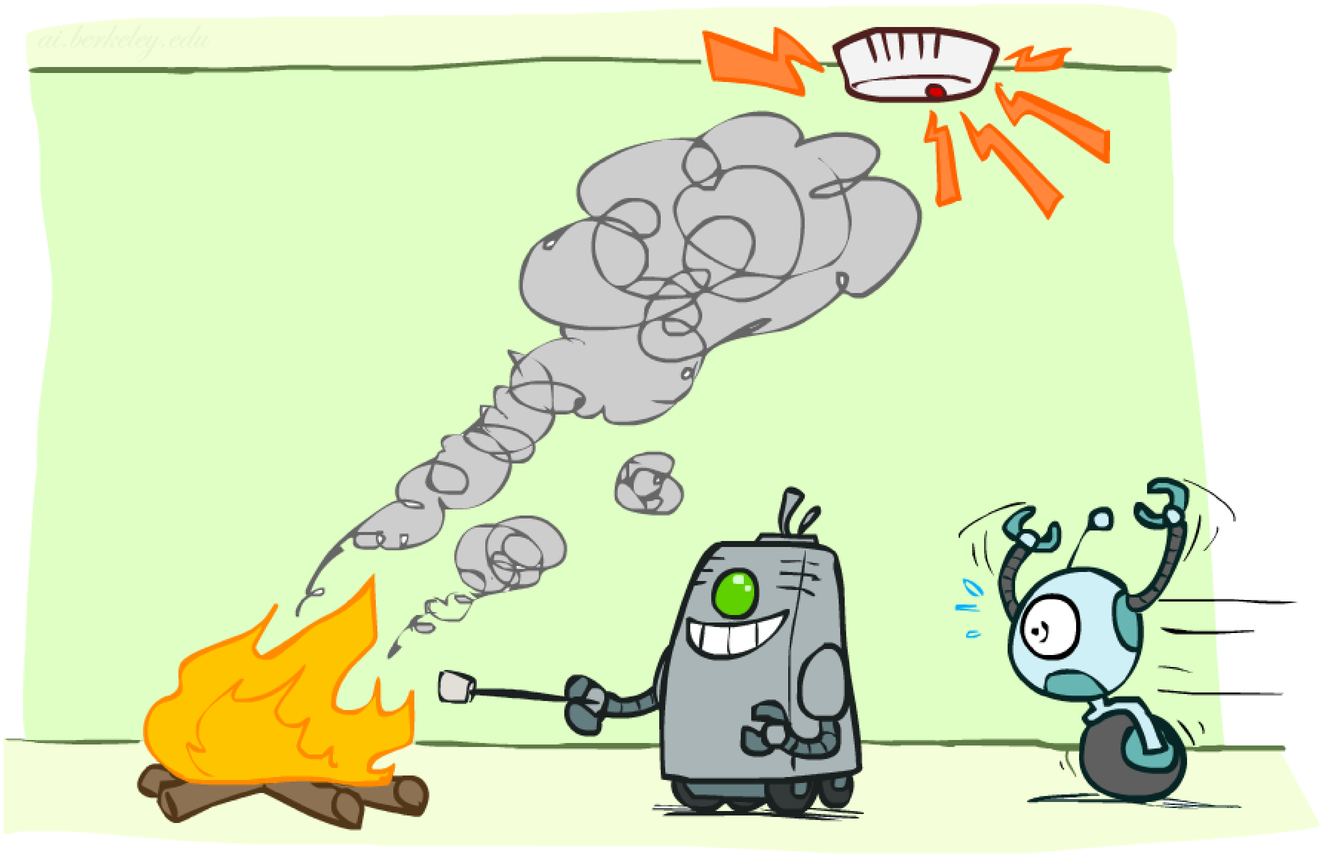


$P(X_1, X_2, \ldots X_n)$

$2^n$



# Conditional Independence



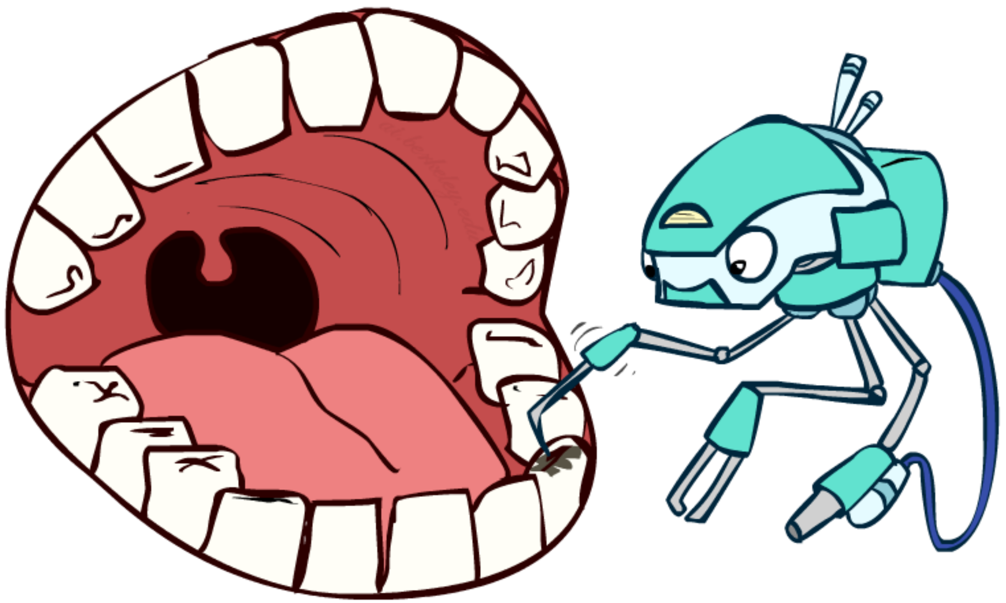
# Conditional Independence

- P(Toothache, Cavity, Catch)

- If I have a cavity, the probability that the probe catches in it doesn't depend on whether I have a toothache:
  - P(+catch | +toothache, +cavity) = P(+catch | +cavity)

- The same independence holds if I don't have a cavity:
  - P(+catch | +toothache, -cavity) = P(+catch| -cavity)

- Catch is *conditionally independent* of Toothache given Cavity:
  - P(Catch | Toothache, Cavity) = P(Catch | Cavity)



- Equivalent statements:
  - P(Toothache | Catch , Cavity) = P(Toothache | Cavity)
  - P(Toothache, Catch | Cavity) = P(Toothache | Cavity) P(Catch | Cavity)
  - One can be derived from the other easily

# Conditional Independence

- Unconditional (absolute) independence very rare (why?)
- *Conditional independence* is our most basic and robust form of knowledge about uncertain environments.
- X is conditionally independent of Y given Z $\quad X \perp\!\!\!\perp Y | Z$

if and only if:

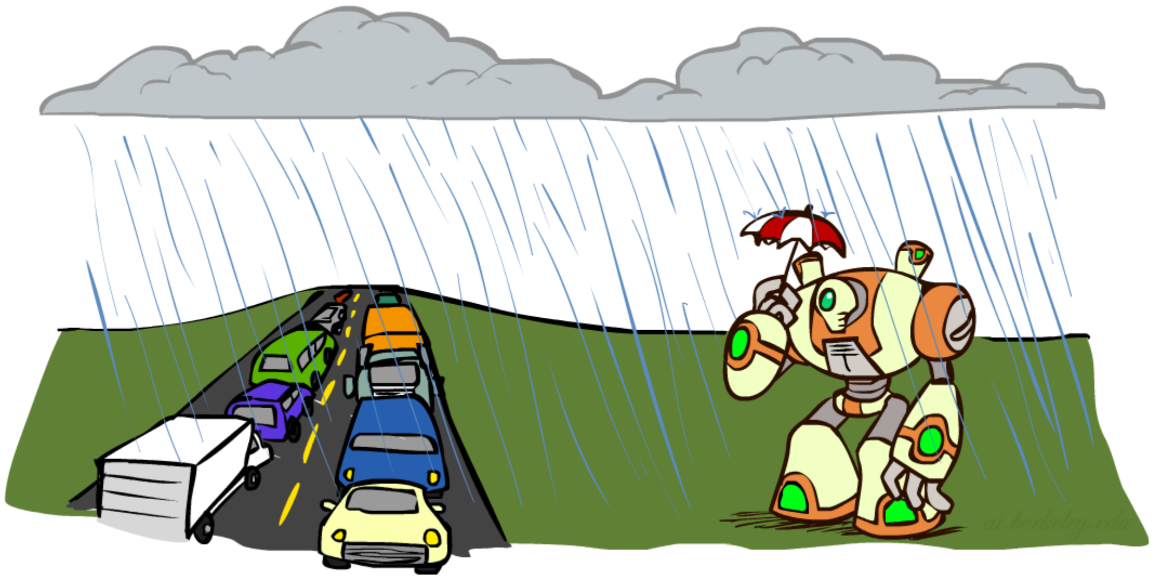
$$\forall x, y, z : P(x, y|z) = P(x|z)P(y|z)$$

or, equivalently, if and only if

$$\forall x, y, z : P(x|z, y) = P(x|z)$$

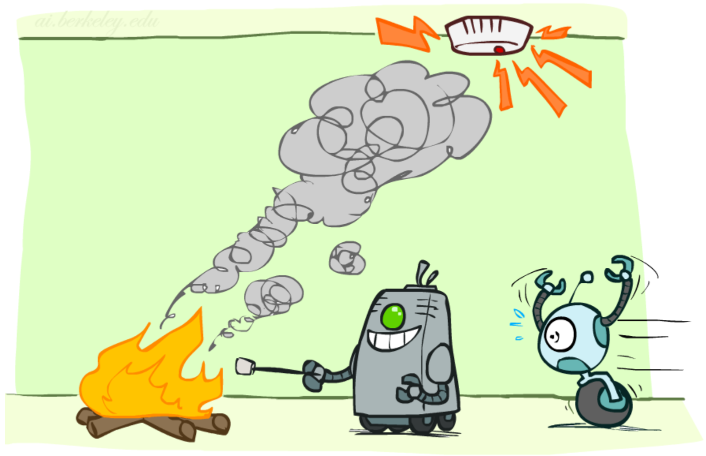
# Conditional Independence

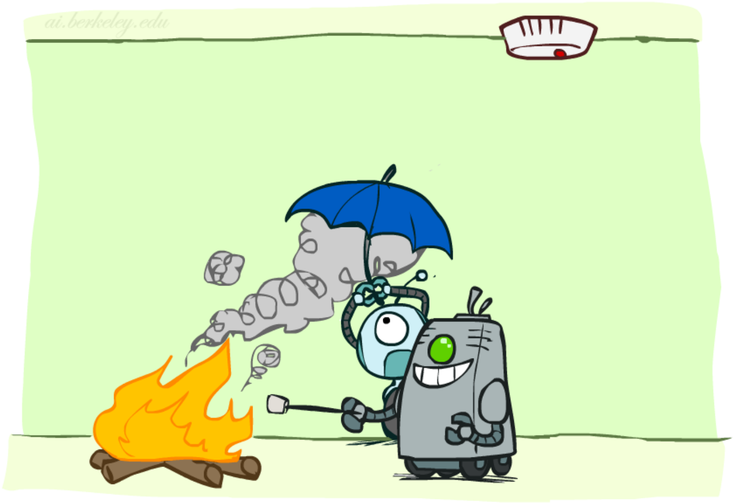
- What about this domain:
  - Traffic
  - Umbrella
  - Raining



# Conditional Independence

- What about this domain:
  - Fire
  - Smoke
  - Alarm

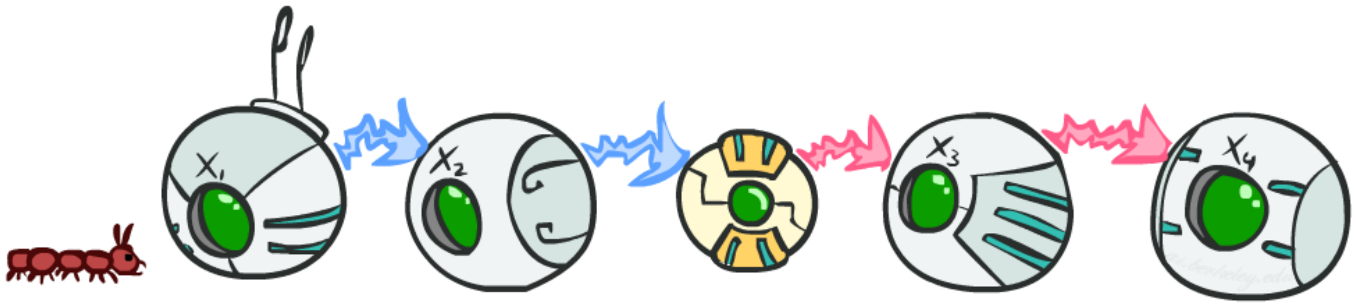




# Probability Recap

- Conditional probability $$P(x|y) = \frac{P(x,y)}{P(y)}$$
- Product rule $$P(x,y) = P(x|y)P(y)$$
- Chain rule
$$\begin{aligned} P(X_1, X_2, \ldots X_n) &= P(X_1)P(X_2|X_1)P(X_3|X_1,X_2)\ldots \\ &= \prod_{i=1}^{n} P(X_i|X_1,\ldots,X_{i-1}) \end{aligned}$$
- X, Y independent if and only if: $\forall x,y : P(x,y) = P(x)P(y)$
- X and Y are conditionally independent given Z if and only if: $X \perp\!\!\!\perp Y | Z$
$$\forall x,y,z : P(x,y|z) = P(x|z)P(y|z)$$

# Markov Models



# Reasoning over Time or Space

- Often, we want to reason about a sequence of observations
  - Speech recognition
  - Robot localization
  - User attention
  - Medical monitoring
- Need to introduce time (or space) into our models

# Markov Models

- Value of X at a given time is called the state

$$X_1 \rightarrow X_2 \rightarrow X_3 \rightarrow X_4 \dashrightarrow$$

$$P(X_1) \qquad P(X_t|X_{t-1})$$

- Parameters: called transition probabilities or dynamics, specify how the state evolves over time (also, initial state probabilities)
- Stationarity assumption: transition probabilities the same at all times
- Same as MDP transition model, but no choice of action

# Joint Distribution of a Markov Model

$$X_1 \rightarrow X_2 \rightarrow X_3 \rightarrow X_4$$

$$P(X_1) \qquad P(X_t|X_{t-1})$$

- Joint distribution:

$$P(X_1, X_2, X_3, X_4) = P(X_1)P(X_2|X_1)P(X_3|X_2)P(X_4|X_3)$$

- More generally:

$$P(X_1, X_2, \ldots, X_T) = P(X_1)P(X_2|X_1)P(X_3|X_2)\ldots P(X_T|X_{T-1})$$

$$= P(X_1)\prod_{t=2}^{T} P(X_t|X_{t-1})$$

- Questions to be resolved:
  - Does this indeed define a joint distribution?
  - Can every joint distribution be factored this way, or are we making some assumptions about the joint distribution by using this factorization?

# Chain Rule and Markov Models

$$X_1 \to X_2 \to X_3 \to X_4$$

- From the chain rule, every joint distribution over $X_1, X_2, X_3, X_4$ can be written as:

$$P(X_1, X_2, X_3, X_4) = P(X_1)P(X_2|X_1)P(X_3|X_1, X_2)P(X_4|X_1, X_2, X_3)$$

- Assuming that

$$X_3 \perp\!\!\!\perp X_1 \mid X_2 \quad \text{and} \quad X_4 \perp\!\!\!\perp X_1, X_2 \mid X_3$$

results in the expression posited on the previous slide:

$$P(X_1, X_2, X_3, X_4) = P(X_1)P(X_2|X_1)P(X_3|X_2)P(X_4|X_3)$$

# Chain Rule and Markov Models

$$X_1 \to X_2 \to X_3 \to X_4 \dashrightarrow$$

- From the chain rule, every joint distribution over $X_1, X_2, \ldots, X_T$ can be written as:

$$P(X_1, X_2, \ldots, X_T) = P(X_1) \prod_{t=2}^{T} P(X_t | X_1, X_2, \ldots, X_{t-1})$$

- Assuming that for all *t*:

$$X_t \perp\!\!\!\perp X_1, \ldots, X_{t-2} \mid X_{t-1}$$

gives us the expression posited on the earlier slide:

$$P(X_1, X_2, \ldots, X_T) = P(X_1) \prod_{t=2}^{T} P(X_t | X_{t-1})$$

# Implied Conditional Independencies

$$X_1 \rightarrow X_2 \rightarrow X_3 \rightarrow X_4$$

- We assumed: $X_3 \perp\!\!\!\perp X_1 \mid X_2$ and $X_4 \perp\!\!\!\perp X_1, X_2 \mid X_3$

- Do we also have $X_1 \perp\!\!\!\perp X_3, X_4 \mid X_2$ ?
  - Yes!
  - Proof:

$$\begin{aligned}
P(X_1 \mid X_2, X_3, X_4) &= \frac{P(X_1, X_2, X_3, X_4)}{P(X_2, X_3, X_4)} \\
&= \frac{P(X_1)P(X_2 \mid X_1)P(X_3 \mid X_2)P(X_4 \mid X_3)}{\sum_{x_1} P(x_1)P(X_2 \mid x_1)P(X_3 \mid X_2)P(X_4 \mid X_3)} \\
&= \frac{P(X_1, X_2)}{P(X_2)} \\
&= P(X_1 \mid X_2)
\end{aligned}$$

# Markov Models Recap

- Explicit assumption for all *t* : $X_t \perp\!\!\!\perp X_1, \ldots, X_{t-2} \mid X_{t-1}$
- Consequence, joint distribution can be written as:

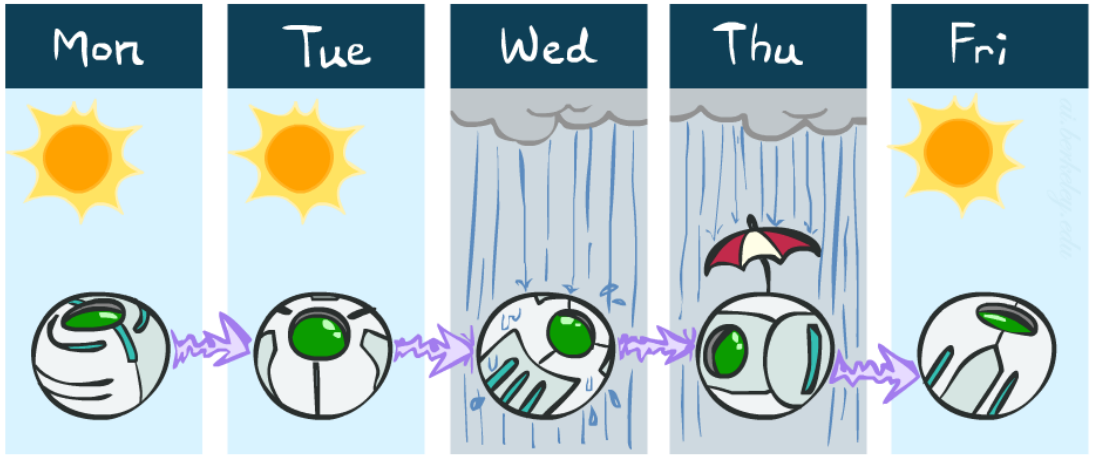
$$P(X_1, X_2, \ldots, X_T) = P(X_1)P(X_2|X_1)P(X_3|X_2)\ldots P(X_T|X_{T-1})$$
$$= P(X_1)\prod_{t=2}^{T} P(X_t|X_{t-1})$$

- Implied conditional independencies: (try to prove this!)
  - Past variables independent of future variables given the present

  i.e., if $t_1 < t_2 < t_3$ or $t_1 > t_2 > t_3$ then: $X_{t_1} \perp\!\!\!\perp X_{t_3} \mid X_{t_2}$
- Additional explicit assumption: $P(X_t \mid X_{t-1})$ is the same for all *t*

# Example Markov Chain: Weather

- States: X = {rain, sun}

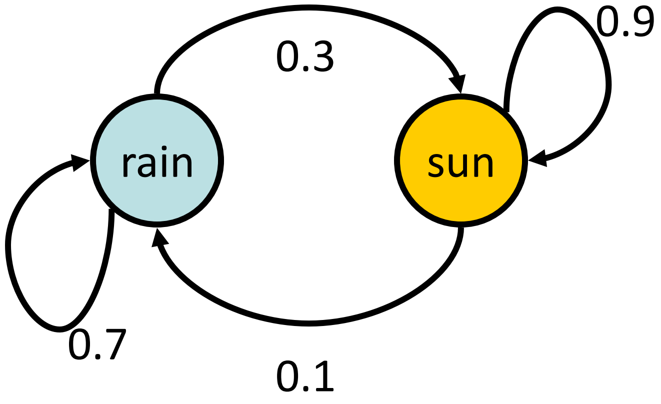


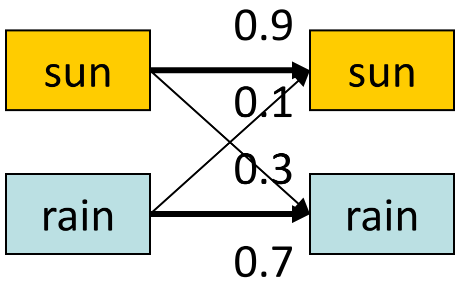

- Initial distribution: 1.0 sun

- CPT $P(X_t \mid X_{t-1})$:

| $X_{t-1}$ | $X_t$ | $P(X_t \mid X_{t-1})$ |
|---|---|---|
| sun | sun | 0.9 |
| sun | rain | 0.1 |
| rain | sun | 0.3 |
| rain | rain | 0.7 |

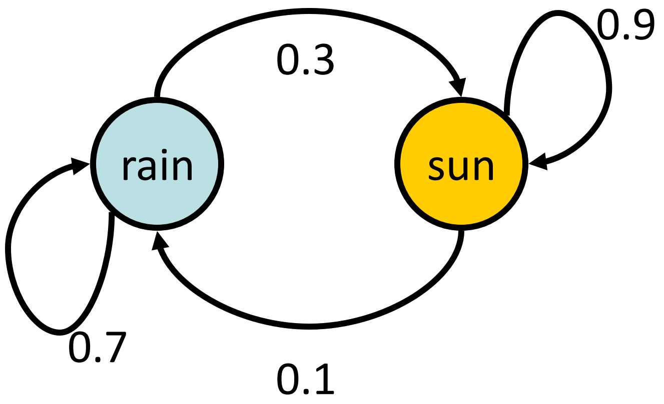
Two new ways of representing the same CPT



# Example Markov Chain: Weather

- Initial distribution: 1.0 sun




- What is the probability distribution after one step?

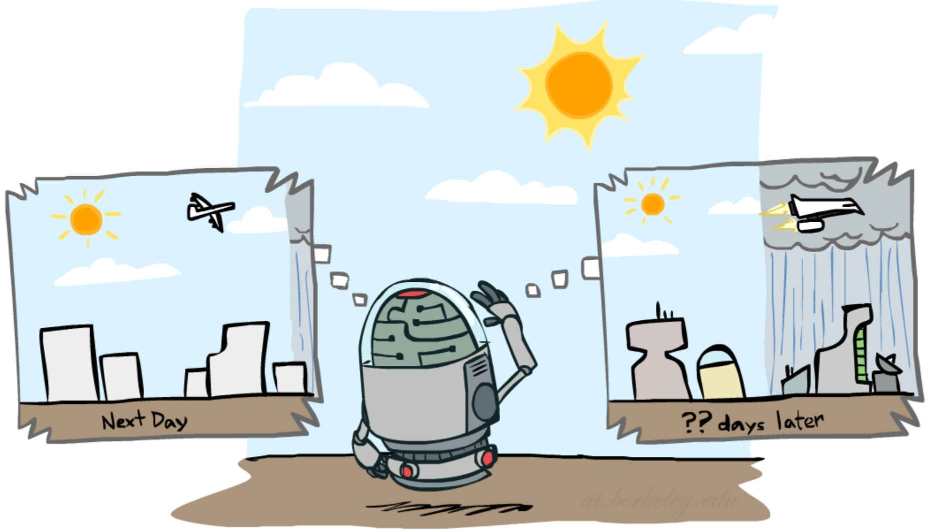
$$P(X_2 = \text{sun}) = \quad P(X_2 = \text{sun} | X_1 = \text{sun}) P(X_1 = \text{sun}) + P(X_2 = \text{sun} | X_1 = \text{rain}) P(X_1 = \text{rain})$$

$$0.9 \cdot 1.0 + 0.3 \cdot 0.0 = 0.9$$

# Mini-Forward Algorithm

- Question: What's P(X) on some day t?

$$X_1 \rightarrow X_2 \rightarrow X_3 \rightarrow X_4 \dashrightarrow$$




$$P(x_1) = \text{known}$$

$$P(x_t) = \sum_{x_{t-1}} P(x_{t-1}, x_t)$$

$$= \sum_{x_{t-1}} P(x_t \mid x_{t-1}) P(x_{t-1})$$

*Forward simulation*

# Example Run of Mini-Forward Algorithm

- From initial observation of sun

$$\underset{P(X_1)}{\left\langle \begin{matrix} 1.0 \\ 0.0 \end{matrix} \right\rangle} \quad \underset{P(X_2)}{\left\langle \begin{matrix} 0.9 \\ 0.1 \end{matrix} \right\rangle} \quad \underset{P(X_3)}{\left\langle \begin{matrix} 0.84 \\ 0.16 \end{matrix} \right\rangle} \quad \underset{P(X_4)}{\left\langle \begin{matrix} 0.804 \\ 0.196 \end{matrix} \right\rangle} \Rightarrow \underset{P(X_\infty)}{\left\langle \begin{matrix} 0.75 \\ 0.25 \end{matrix} \right\rangle}$$

- From initial observation of rain

$$\underset{P(X_1)}{\left\langle \begin{matrix} 0.0 \\ 1.0 \end{matrix} \right\rangle} \quad \underset{P(X_2)}{\left\langle \begin{matrix} 0.3 \\ 0.7 \end{matrix} \right\rangle} \quad \underset{P(X_3)}{\left\langle \begin{matrix} 0.48 \\ 0.52 \end{matrix} \right\rangle} \quad \underset{P(X_4)}{\left\langle \begin{matrix} 0.588 \\ 0.412 \end{matrix} \right\rangle} \Rightarrow \underset{P(X_\infty)}{\left\langle \begin{matrix} 0.75 \\ 0.25 \end{matrix} \right\rangle}$$

- From yet another initial distribution $P(X_1)$:

$$\underset{P(X_1)}{\left\langle \begin{matrix} p \\ 1-p \end{matrix} \right\rangle} \quad \cdots \Rightarrow \underset{P(X_\infty)}{\left\langle \begin{matrix} 0.75 \\ 0.25 \end{matrix} \right\rangle}$$

[Demo: L13D1,2,3]

# Stationary Distributions

- For most chains:
  - Influence of the initial distribution gets less and less over time.
  - The distribution we end up in is independent of the initial distribution

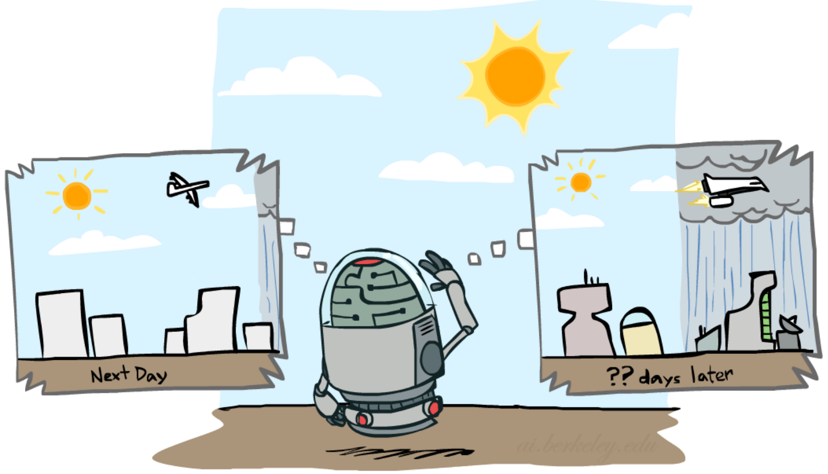
- Stationary distribution:
  - The distribution we end up with is called the stationary distribution $P_\infty$ of the chain
  - It satisfies

$$P_\infty(X) = P_{\infty+1}(X) = \sum_x P(X|x) P_\infty(x)$$

# Example: Stationary Distributions

- Question: What's P(X) at time t = infinity?

$$X_1 \rightarrow X_2 \rightarrow X_3 \rightarrow X_4 \dashrightarrow$$




$$P_\infty(sun) = P(sun|sun)P_\infty(sun) + P(sun|rain)P_\infty(rain)$$
$$P_\infty(rain) = P(rain|sun)P_\infty(sun) + P(rain|rain)P_\infty(rain)$$

$$P_\infty(sun) = 0.9P_\infty(sun) + 0.3P_\infty(rain)$$
$$P_\infty(rain) = 0.1P_\infty(sun) + 0.7P_\infty(rain)$$

$$P_\infty(sun) = 3P_\infty(rain)$$
$$P_\infty(rain) = 1/3P_\infty(sun)$$

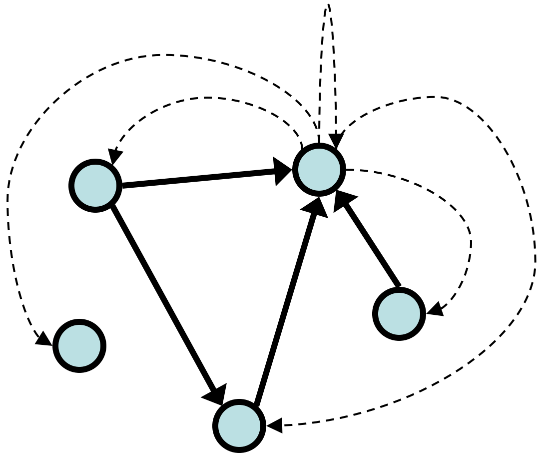
| $X_{t-1}$ | $X_t$ | $P(X_t\|X_{t-1})$ |
|---|---|---|
| sun | sun | 0.9 |
| sun | rain | 0.1 |
| rain | sun | 0.3 |
| rain | rain | 0.7 |

Also: $P_\infty(sun) + P_\infty(rain) = 1$

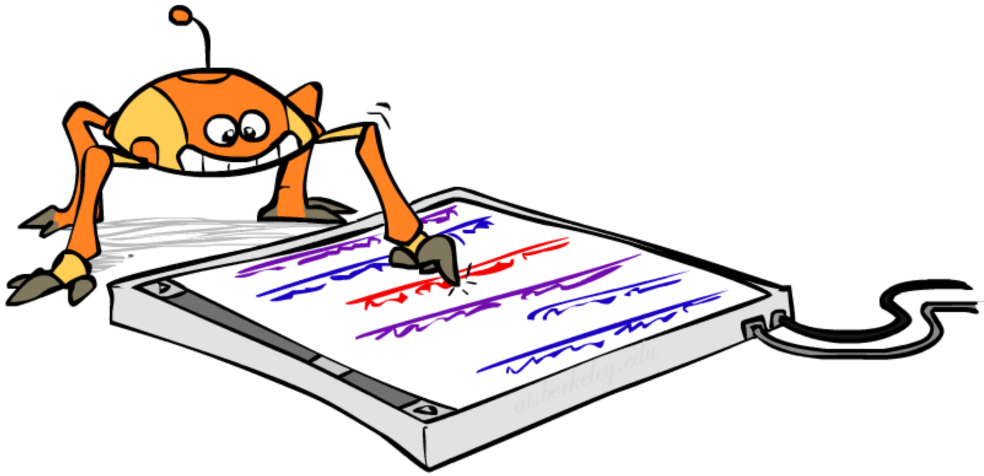
$$P_\infty(sun) = 3/4$$
$$P_\infty(rain) = 1/4$$

# Application of Stationary Distribution: Web Link Analysis

- PageRank over a web graph
  - Each web page is a state
  - Initial distribution: uniform over pages
  - Transitions:
    - With prob. c, uniform jump to a random page (dotted lines, not all shown)
    - With prob. 1-c, follow a random outlink (solid lines)



- Stationary distribution
  - Will spend more time on highly reachable pages
  - E.g. many ways to get to the Acrobat Reader download page
  - Somewhat robust to link spam
  - Google 1.0 returned the set of pages containing all your keywords in decreasing rank, now all search engines use link analysis along with many other factors (rank actually getting less important over time)



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