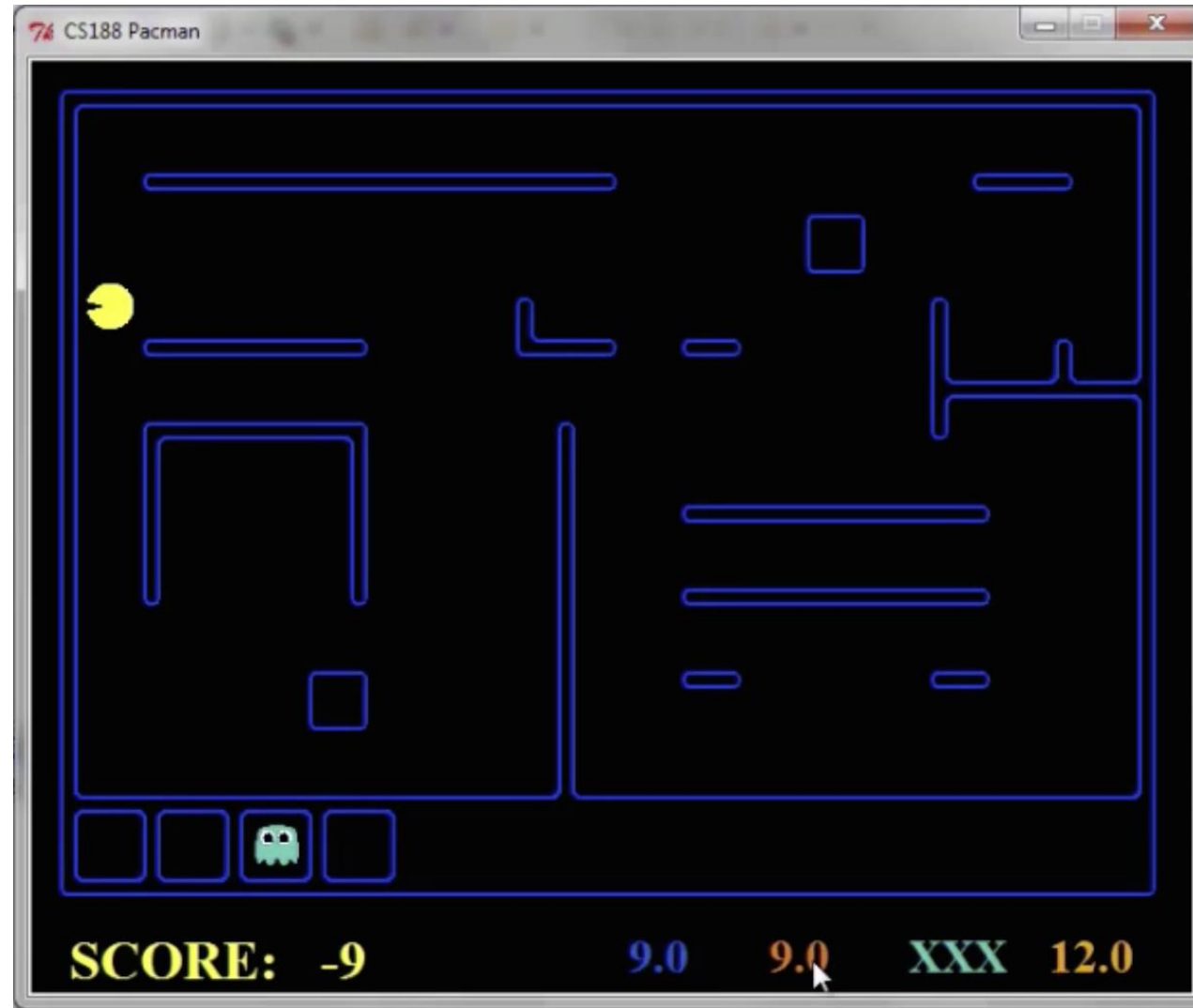


# Hidden Markov Models (HMMs)



# Pacman – Sonar (P4)

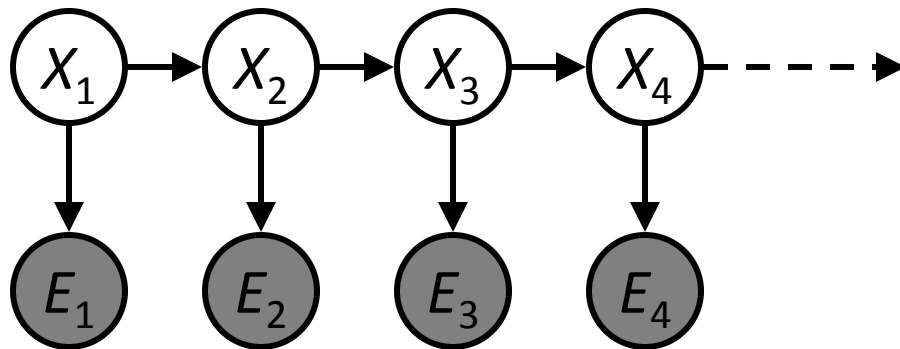


# Probability Recap

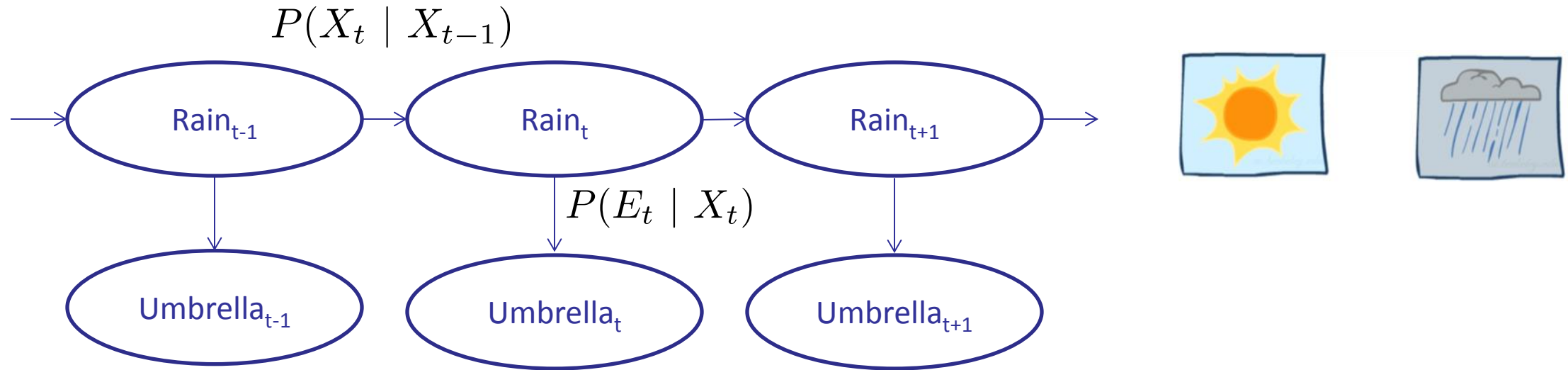
- Conditional probability  $P(x|y) = \frac{P(x, y)}{P(y)}$
- Product rule  $P(x, y) = P(x|y)P(y)$
- Chain rule 
$$\begin{aligned} P(X_1, X_2, \dots, X_n) &= P(X_1)P(X_2|X_1)P(X_3|X_1, X_2) \dots \\ &= \prod_{i=1}^n P(X_i|X_1, \dots, X_{i-1}) \end{aligned}$$
- X, Y independent if and only if:  $\forall x, y : P(x, y) = P(x)P(y)$
- X and Y are conditionally independent given Z if and only if:  $X \perp\!\!\!\perp Y | Z$   
 $\forall x, y, z : P(x, y|z) = P(x|z)P(y|z)$

# Hidden Markov Models

- Markov chains not so useful for most agents
  - Need observations to update your beliefs
- Hidden Markov models (HMMs)
  - Underlying Markov chain over states  $X$
  - You observe outputs (effects) at each time step



# Example: Weather HMM



- An HMM is defined by:

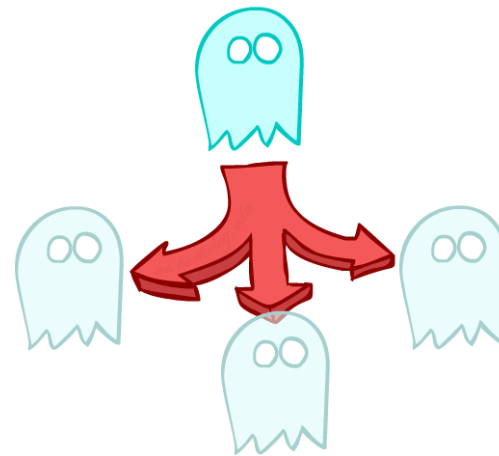
- Initial distribution:  $P(X_1)$
  - Transitions:  $P(X_t | X_{t-1})$
  - Emissions:  $P(E_t | X_t)$

$R_t$	$R_{t+1}$	$P(R_{t+1}   R_t)$
+r	+r	0.7
+r	-r	0.3
-r	+r	0.3
-r	-r	0.7

$R_t$	$U_t$	$P(U_t   R_t)$
+r	+u	0.9
+r	-u	0.1
-r	+u	0.2
-r	-u	0.8

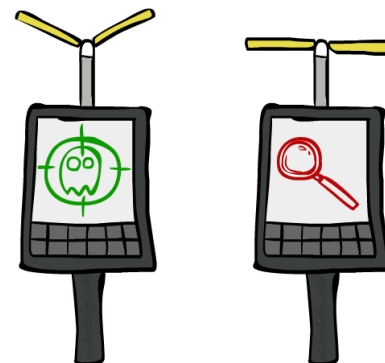
# Example: Ghostbusters HMM

- $P(X_1)$  = uniform
- $P(X|X')$  = usually move clockwise, but sometimes move in a random direction or stay in place
- $P(R_{ij} | X)$  = same sensor model as before: red means close, green means far away.



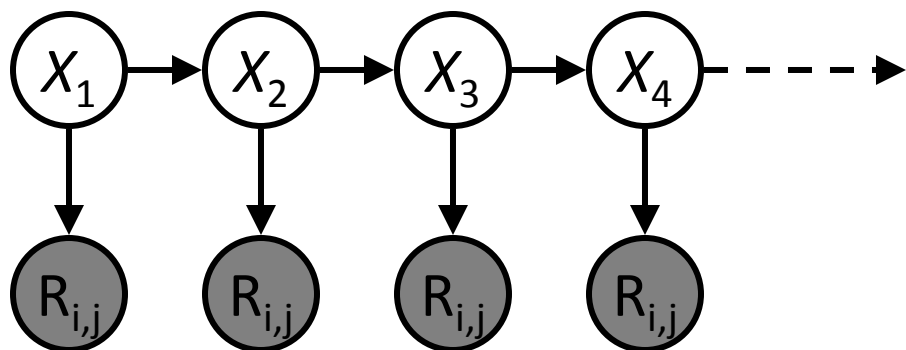
1/9	1/9	1/9
1/9	1/9	1/9
1/9	1/9	1/9

$P(X_1)$

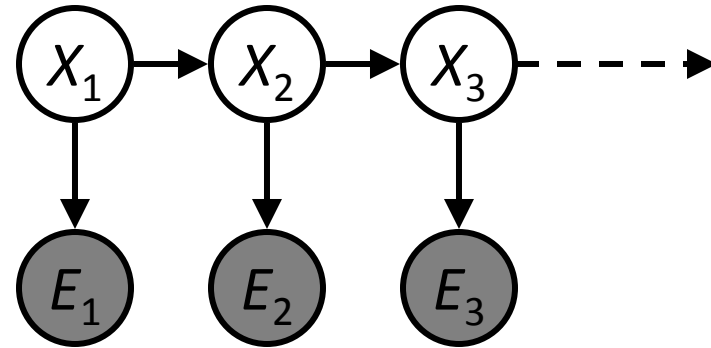


1/6	1/6	1/2
0	1/6	0
0	0	0

$P(X|X' = \langle 1, 2 \rangle)$



# Joint Distribution of an HMM



- Joint distribution:

$$P(X_1, E_1, X_2, E_2, X_3, E_3) = P(X_1)P(E_1|X_1)P(X_2|X_1)P(E_2|X_2)P(X_3|X_2)P(E_3|X_3)$$

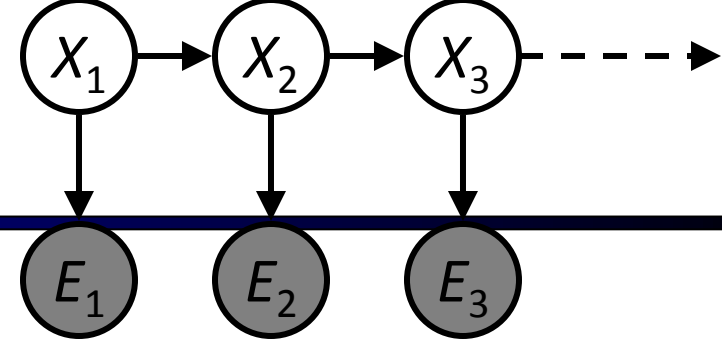
- More generally:

$$P(X_1, E_1, \dots, X_T, E_T) = P(X_1)P(E_1|X_1) \prod_{t=2}^T P(X_t|X_{t-1})P(E_t|X_t)$$

- Questions to be resolved:

- Does this indeed define a joint distribution?
- Can every joint distribution be factored this way, or are we making some assumptions about the joint distribution by using this factorization?

# Chain Rule and HMMs



- From the chain rule, *every* joint distribution over  $X_1, E_1, X_2, E_2, X_3, E_3$  can be written as:

$$P(X_1, E_1, X_2, E_2, X_3, E_3) = P(X_1)P(E_1|X_1)P(X_2|X_1, E_1)P(E_2|X_1, E_1, X_2) \\ P(X_3|X_1, E_1, X_2, E_2)P(E_3|X_1, E_1, X_2, E_2, X_3)$$

- Assuming that

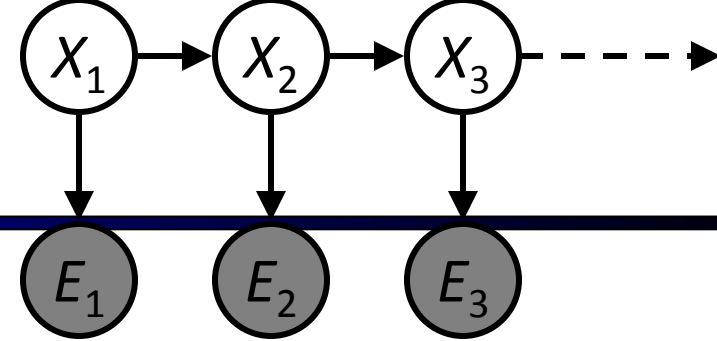
$$X_2 \perp\!\!\!\perp E_1 \mid X_1, \quad E_2 \perp\!\!\!\perp X_1, E_1 \mid X_2, \quad X_3 \perp\!\!\!\perp X_1, E_1, E_2 \mid X_2, \quad E_3 \perp\!\!\!\perp X_1, E_1, X_2, E_2 \mid X_3$$

gives us the expression posited on the previous slide:

$$P(X_1, E_1, X_2, E_2, X_3, E_3) = P(X_1)P(E_1|X_1)P(X_2|X_1)P(E_2|X_2)P(X_3|X_2)P(E_3|X_3)$$



# Chain Rule and HMMs



- From the chain rule, every joint distribution over  $X_1, E_1, \dots, X_T, E_T$  can be written as:

$$P(X_1, E_1, \dots, X_T, E_T) = P(X_1)P(E_1|X_1) \prod_{t=2}^T P(X_t|X_1, E_1, \dots, X_{t-1}, E_{t-1})P(E_t|X_1, E_1, \dots, X_{t-1}, E_{t-1}, X_t)$$

- Assuming that for all  $t$ :

- State independent of all past states and all past evidence given the previous state, i.e.:

$$X_t \perp\!\!\!\perp X_1, E_1, \dots, X_{t-2}, E_{t-2}, E_{t-1} \mid X_{t-1}$$

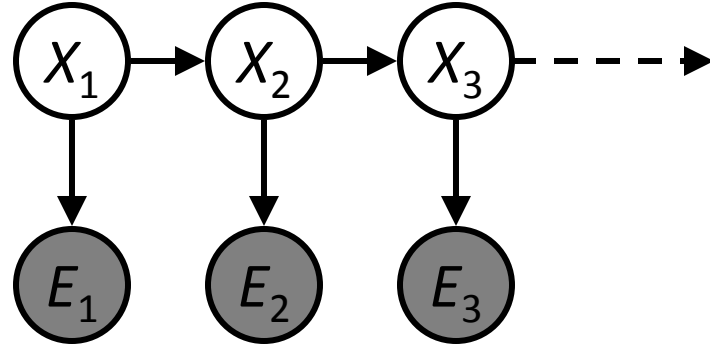
- Evidence is independent of all past states and all past evidence given the current state, i.e.:

$$E_t \perp\!\!\!\perp X_1, E_1, \dots, X_{t-2}, E_{t-2}, X_{t-1}, E_{t-1} \mid X_t$$

gives us the expression posited on the earlier slide:

$$P(X_1, E_1, \dots, X_T, E_T) = P(X_1)P(E_1|X_1) \prod_{t=2}^T P(X_t|X_{t-1})P(E_t|X_t)$$

# Implied Conditional Independencies



- Many implied conditional independencies, e.g.,

$$E_1 \perp\!\!\!\perp X_2, E_2, X_3, E_3 \mid X_1$$

- To prove them

- Approach 1: follow similar (algebraic) approach to what we did in the Markov models lecture
- Approach 2: directly from the graph structure (3 lectures from now)

- Intuition: If path between U and V goes through W, then  $U \perp\!\!\!\perp V \mid W$  [Some fineprint later]

# Real HMM Examples

---

- **Speech recognition HMMs:**
  - Observations are acoustic signals (continuous valued)
  - States are specific positions in specific words (so, tens of thousands)
- **Machine translation HMMs:**
  - Observations are words (tens of thousands)
  - States are translation options
- **Robot tracking:**
  - Observations are range readings (continuous)
  - States are positions on a map (continuous)

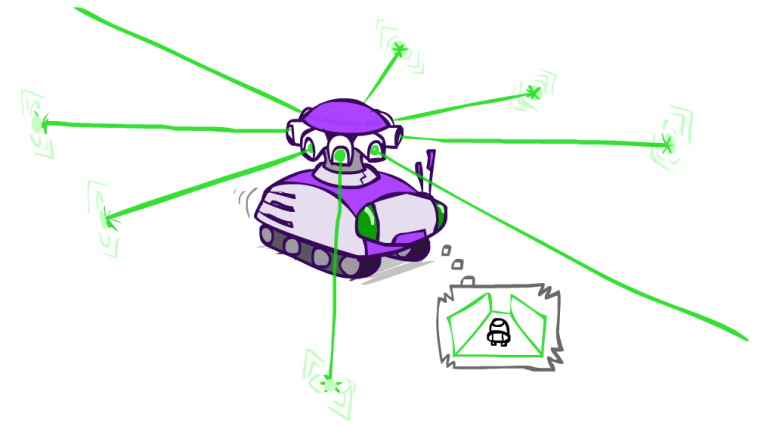
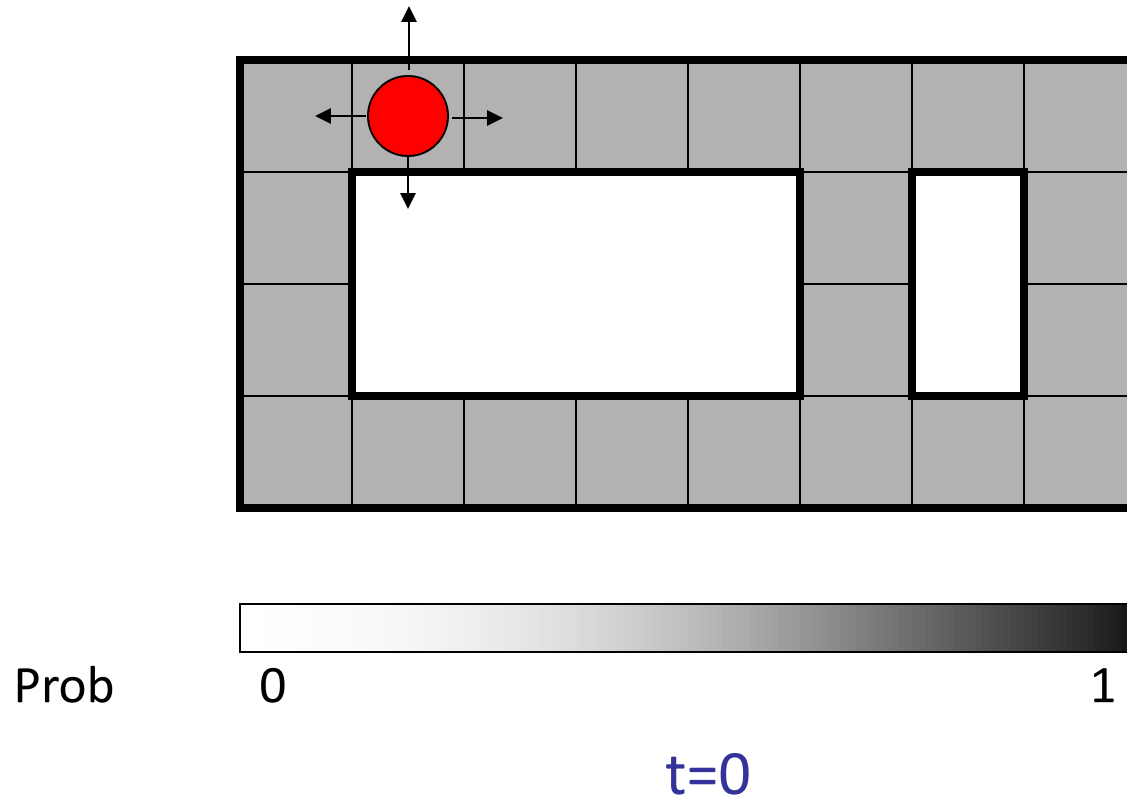
# Filtering / Monitoring

---

- Filtering, or monitoring, is the task of tracking the distribution  $B_t(X) = P_t(X_t | e_1, \dots, e_t)$  (the belief state) over time
- We start with  $B_1(X)$  in an initial setting, usually uniform
- As time passes, or we get observations, we update  $B(X)$
- The Kalman filter was invented in the 60's and first implemented as a method of trajectory estimation for the Apollo program

# Example: Robot Localization

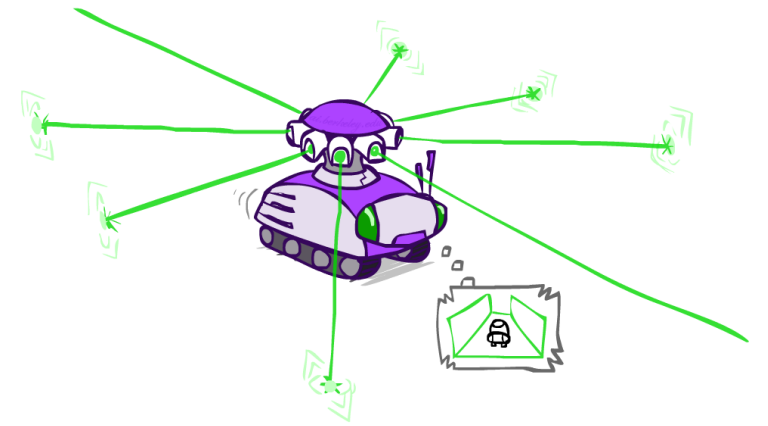
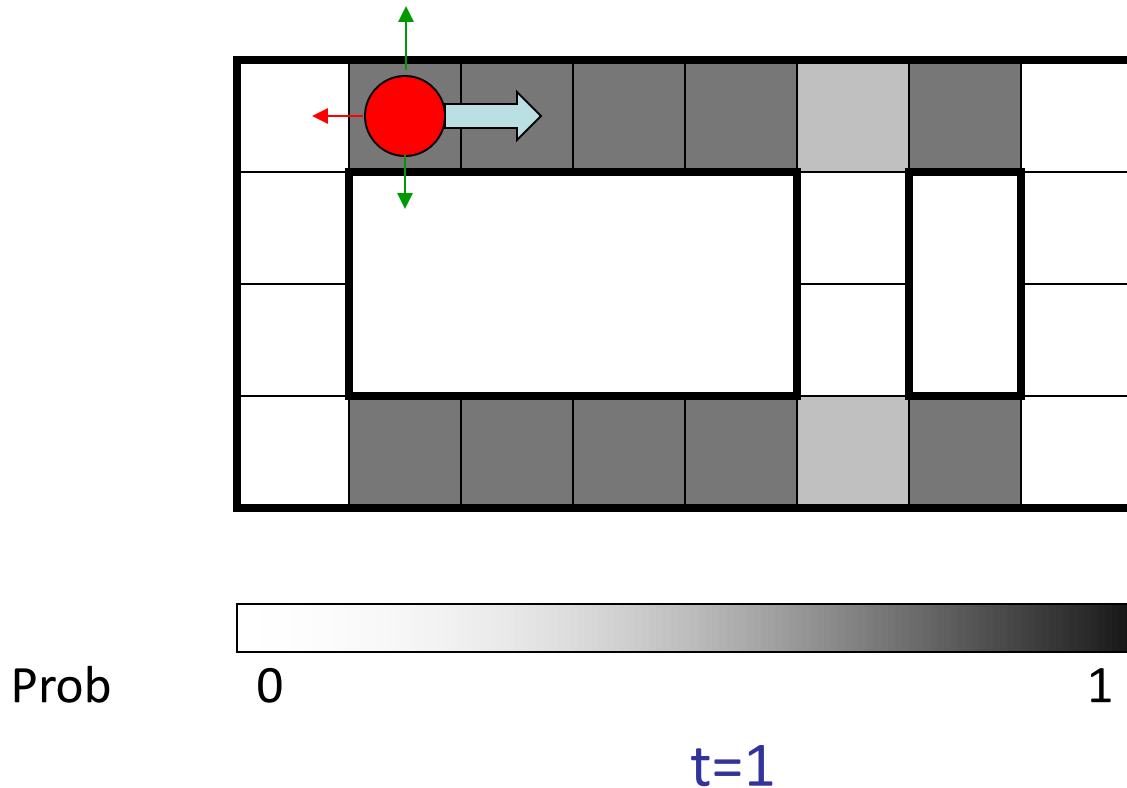
Example from  
Michael Pfeiffer



Sensor model: can read in which directions there is a wall,  
never more than 1 mistake

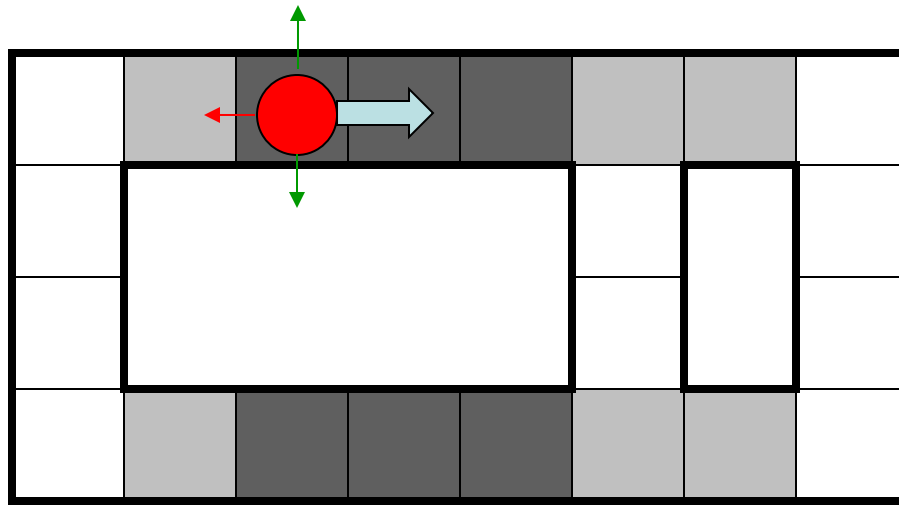
Motion model: may not execute action with small prob.

# Example: Robot Localization



Lighter grey: was possible to get the reading, but less likely b/c required 1 mistake

# Example: Robot Localization



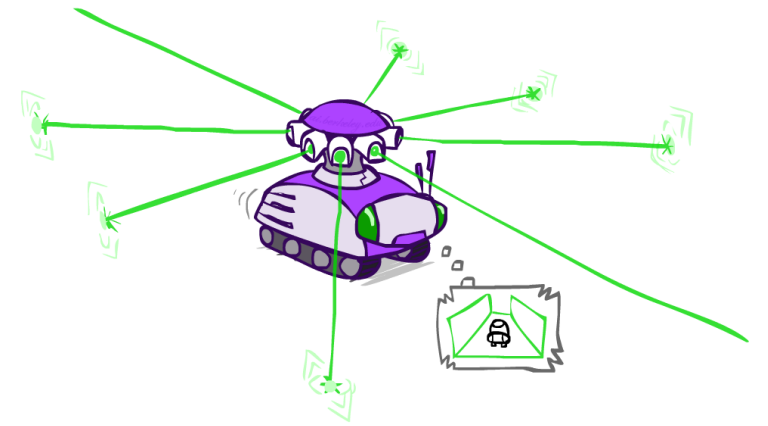
Prob



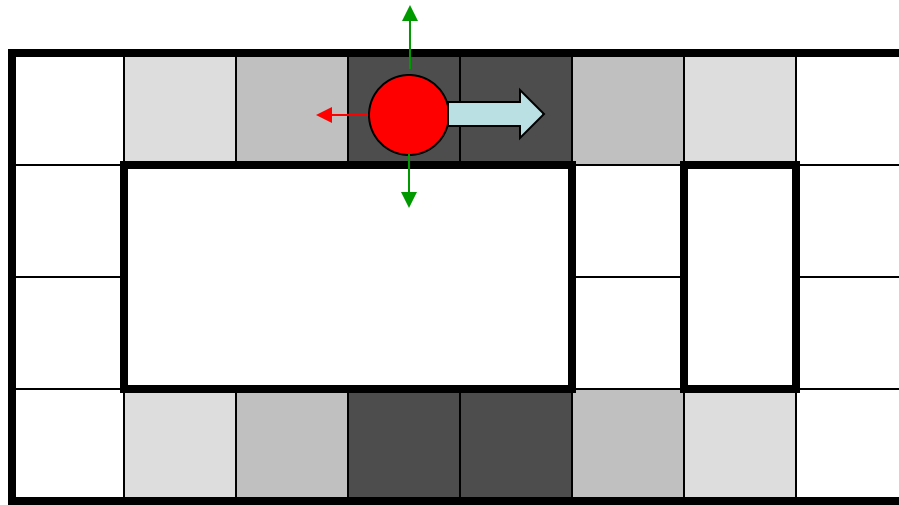
0

1

t=2



# Example: Robot Localization

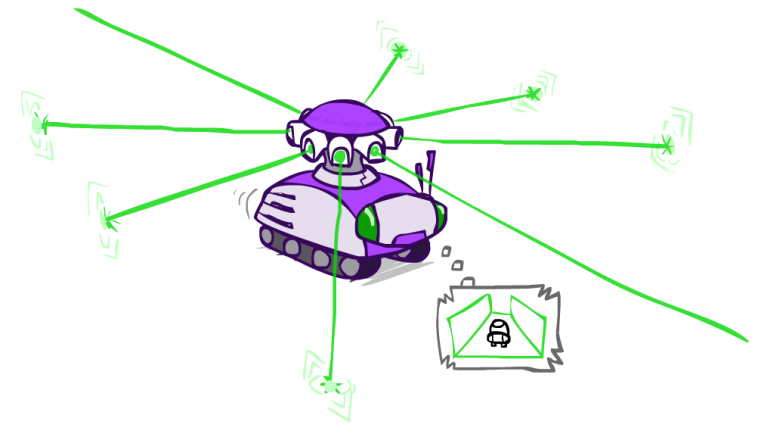


Prob

0

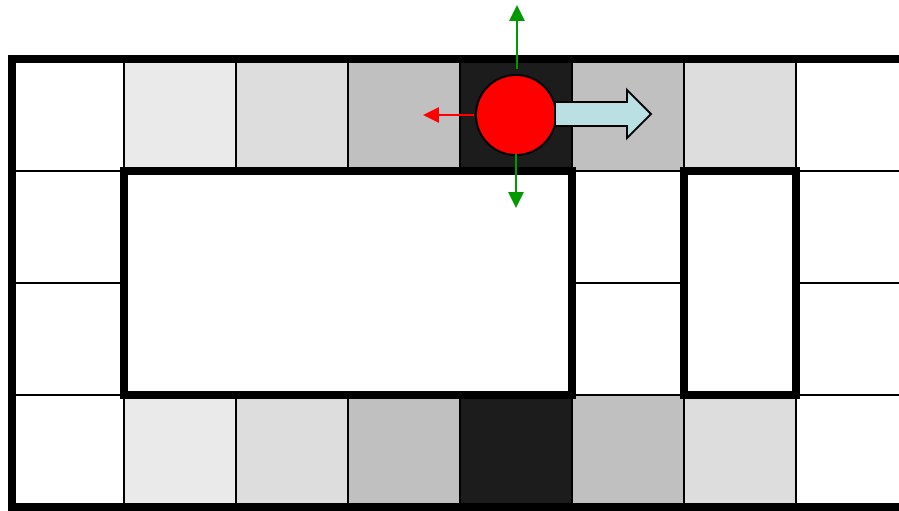
1

t=3





# Example: Robot Localization

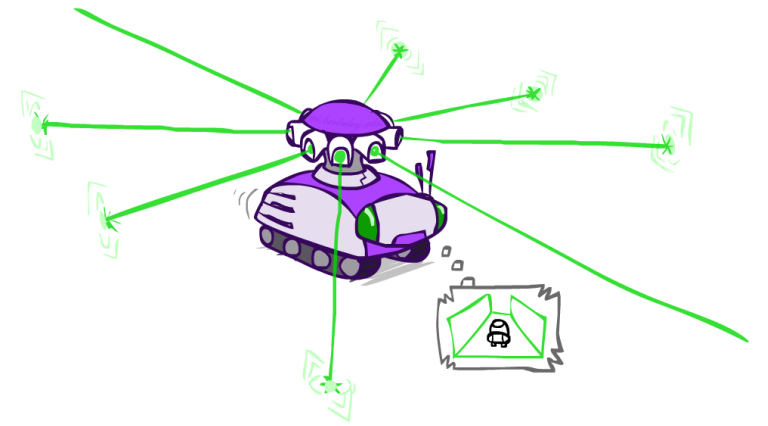


Prob

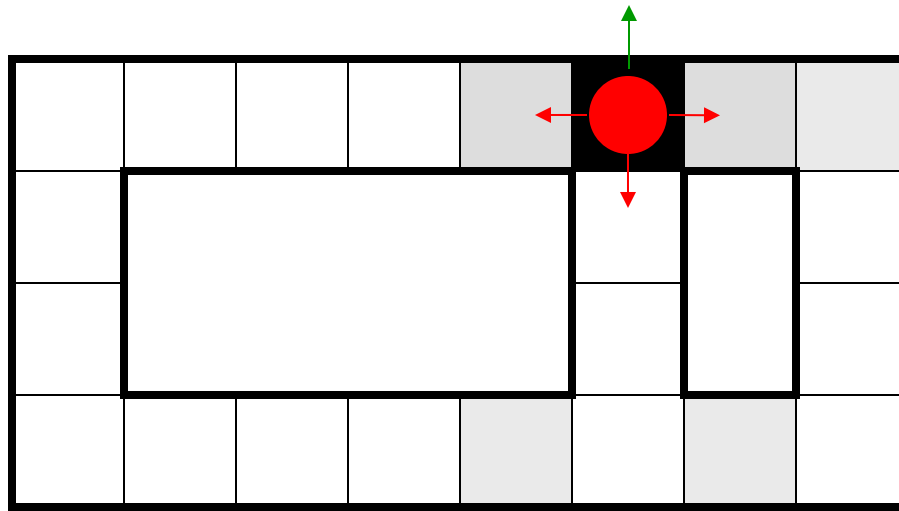
0

1

$t=4$



# Example: Robot Localization

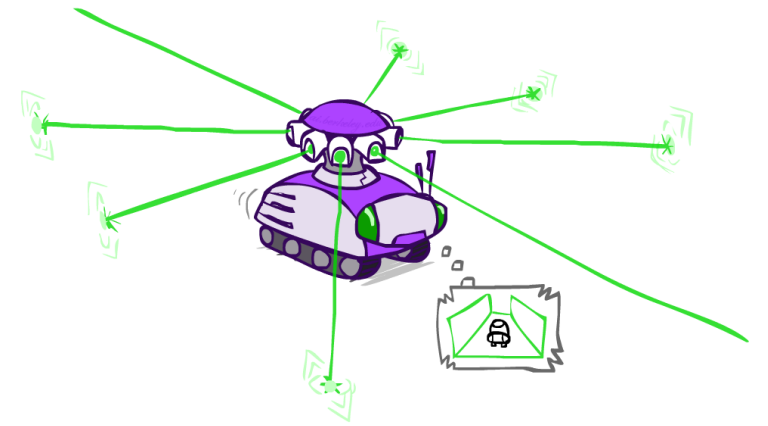


Prob

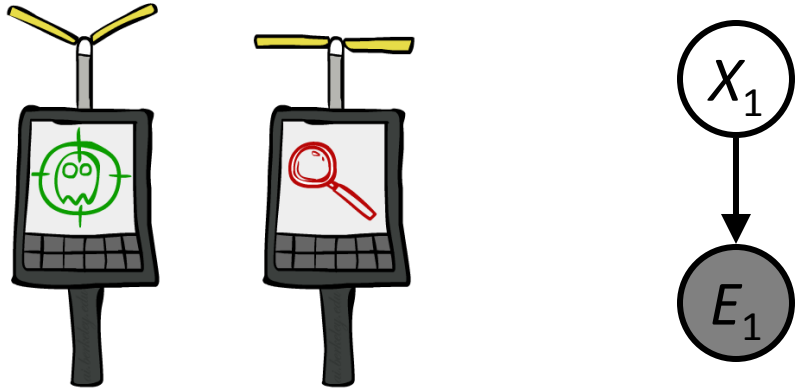
0

1

t=5

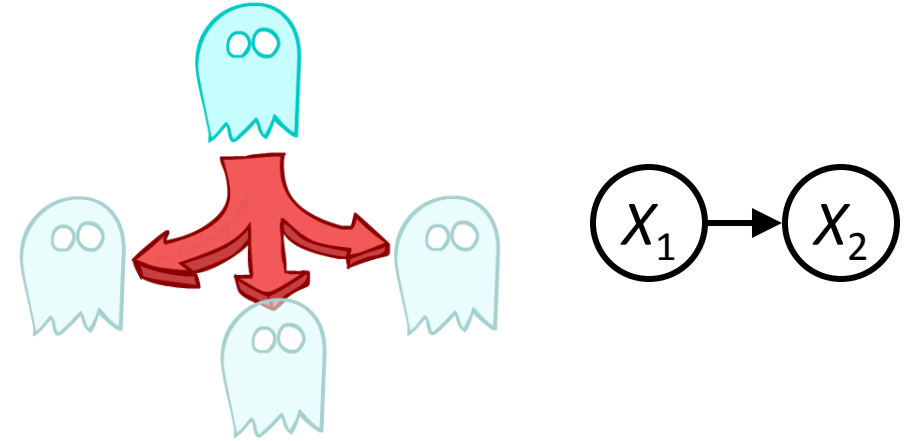


# Inference: Base Cases



$$P(X_1|e_1)$$

$$\begin{aligned} P(x_1|e_1) &= P(x_1, e_1)/P(e_1) \\ &\propto_{X_1} P(x_1, e_1) \\ &= P(x_1)P(e_1|x_1) \end{aligned}$$



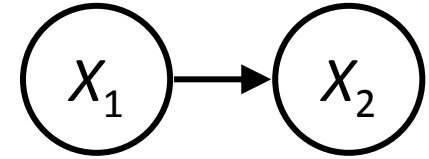
$$P(X_2)$$

$$\begin{aligned} P(x_2) &= \sum_{x_1} P(x_1, x_2) \\ &= \sum_{x_1} P(x_1)P(x_2|x_1) \end{aligned}$$

# Passage of Time

- Assume we have current belief  $P(X \mid \text{evidence to date})$

$$B(X_t) = P(X_t | e_{1:t})$$



- Then, after one time step passes:

$$\begin{aligned} P(X_{t+1} | e_{1:t}) &= \sum_{x_t} P(X_{t+1}, x_t | e_{1:t}) \\ &= \sum_{x_t} P(X_{t+1} | x_t, e_{1:t}) P(x_t | e_{1:t}) \\ &= \sum_{x_t} P(X_{t+1} | x_t) P(x_t | e_{1:t}) \end{aligned}$$

- Or compactly:

$$B'(X_{t+1}) = \sum_{x_t} P(X' | x_t) B(x_t)$$

- Basic idea: beliefs get “pushed” through the transitions
  - With the “B” notation, we have to be careful about what time step  $t$  the belief is about, and what evidence it includes

# Example: Passage of Time

- As time passes, uncertainty “accumulates”

(Transition model: ghosts usually go clockwise)

<0.01	<0.01	<0.01	<0.01	<0.01	<0.01
<0.01	<0.01	<0.01	<0.01	<0.01	<0.01
<0.01	<0.01	1.00	<0.01	<0.01	<0.01
<0.01	<0.01	<0.01	<0.01	<0.01	<0.01

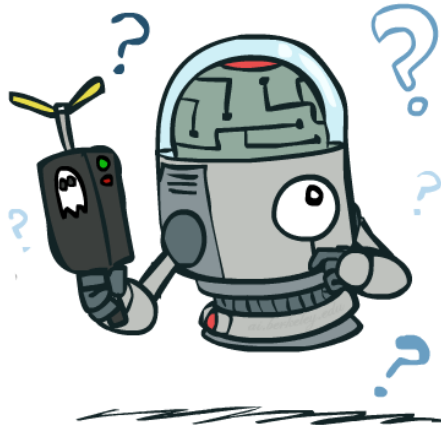
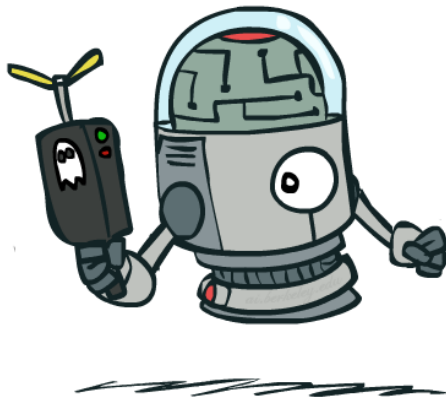
T = 1

<0.01	<0.01	<0.01	<0.01	<0.01	<0.01
<0.01	<0.01	0.06	<0.01	<0.01	<0.01
<0.01	0.76	0.06	0.06	<0.01	<0.01
<0.01	<0.01	0.06	<0.01	<0.01	<0.01

T = 2

0.05	0.01	0.05	<0.01	<0.01	<0.01
0.02	0.14	0.11	0.35	<0.01	<0.01
0.07	0.03	0.05	<0.01	0.03	<0.01
0.03	0.03	<0.01	<0.01	<0.01	<0.01

T = 5



# Observation

- Assume we have current belief  $P(X \mid \text{previous evidence})$ :

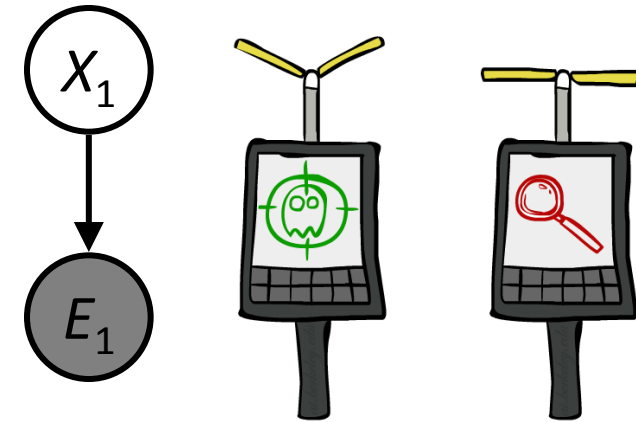
$$B'(X_{t+1}) = P(X_{t+1} | e_{1:t})$$

- Then, after evidence comes in:

$$\begin{aligned} P(X_{t+1} | e_{1:t+1}) &= P(X_{t+1}, e_{t+1} | e_{1:t}) / P(e_{t+1} | e_{1:t}) \\ &\propto_{X_{t+1}} P(X_{t+1}, e_{t+1} | e_{1:t}) \\ &= P(e_{t+1} | e_{1:t}, X_{t+1}) P(X_{t+1} | e_{1:t}) \\ &= P(e_{t+1} | X_{t+1}) P(X_{t+1} | e_{1:t}) \end{aligned}$$

- Or, compactly:

$$B(X_{t+1}) \propto_{X_{t+1}} P(e_{t+1} | X_{t+1}) B'(X_{t+1})$$



- Basic idea: beliefs “reweighted” by likelihood of evidence
- Unlike passage of time, we have to renormalize

# Example: Observation

- As we get observations, beliefs get reweighted, uncertainty “decreases”

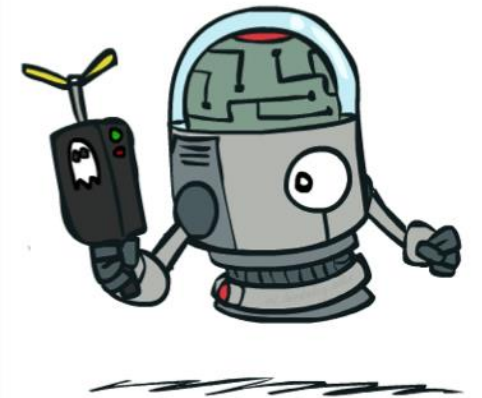
0.05	0.01	0.05	<0.01	<0.01	<0.01
0.02	0.14	0.11	0.35	<0.01	<0.01
0.07	0.03	0.05	<0.01	0.03	<0.01
0.03	0.03	<0.01	<0.01	<0.01	<0.01

Before observation

<0.01	<0.01	<0.01	<0.01	0.02	<0.01
<0.01	<0.01	<0.01	0.83	0.02	<0.01
<0.01	<0.01	0.11	<0.01	<0.01	<0.01
<0.01	<0.01	<0.01	<0.01	<0.01	<0.01

After observation

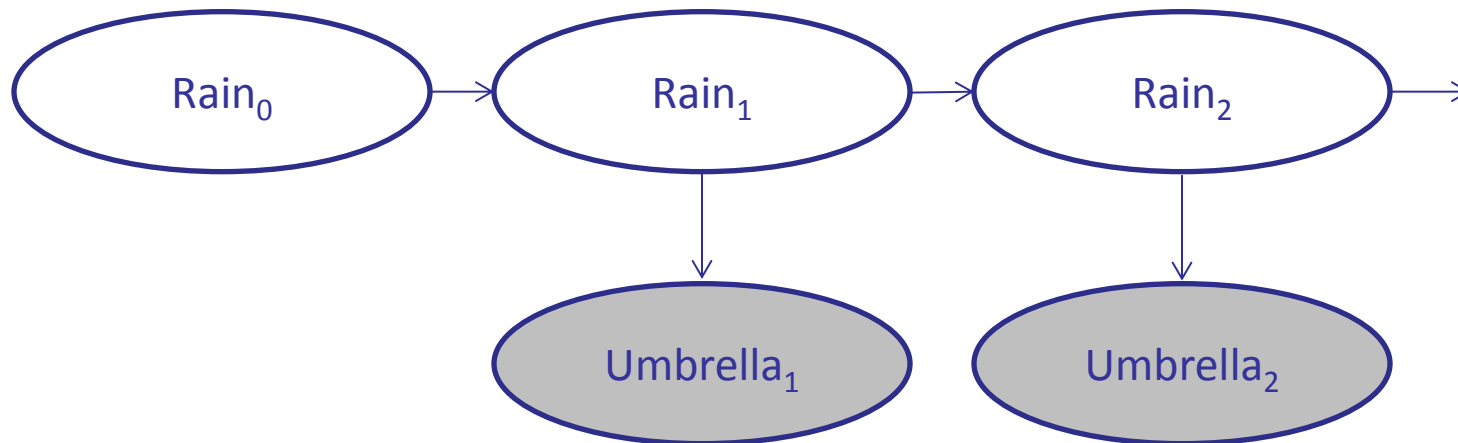
$$B(X) \propto P(e|X)B'(X)$$



# Example: Weather HMM



$$\begin{array}{l}
 B(+r) = 0.5 \\
 B(-r) = 0.5
 \end{array}
 \begin{array}{l}
 \nearrow \\
 \downarrow \\
 \nearrow
 \end{array}
 \begin{array}{l}
 B'(+r) = 0.5 \\
 B'(-r) = 0.5 \\
 \\
 B(+r) = 0.818 \\
 B(-r) = 0.182
 \end{array}
 \begin{array}{l}
 \nearrow \\
 \downarrow \\
 \nearrow
 \end{array}
 \begin{array}{l}
 B'(+r) = 0.627 \\
 B'(-r) = 0.373 \\
 \\
 B(+r) = 0.883 \\
 B(-r) = 0.117
 \end{array}$$



$R_t$	$R_{t+1}$	$P(R_{t+1} R_t)$
+r	+r	0.7
+r	-r	0.3
-r	+r	0.3
-r	-r	0.7

$R_t$	$U_t$	$P(U_t R_t)$
+r	+u	0.9
+r	-u	0.1
-r	+u	0.2
-r	-u	0.8



# The Forward Algorithm

- We are given evidence at each time and want to know

$$B_t(X) = P(X_t|e_{1:t})$$

- We can derive the following updates

$$\begin{aligned} P(x_t|e_{1:t}) &\propto_X P(x_t, e_{1:t}) \\ &= \sum_{x_{t-1}} P(x_{t-1}, x_t, e_{1:t}) \\ &= \sum_{x_{t-1}} P(x_{t-1}, e_{1:t-1}) P(x_t|x_{t-1}) P(e_t|x_t) \\ &= P(e_t|x_t) \sum_{x_{t-1}} P(x_t|x_{t-1}) P(x_{t-1}, e_{1:t-1}) \end{aligned}$$

We can normalize as we go if we want to have  $P(x|e)$  at each time step, or just once at the end...

# Online Belief Updates

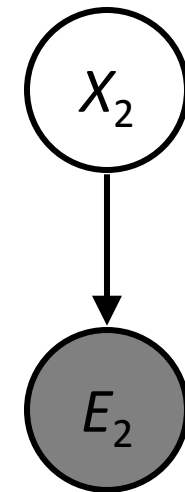
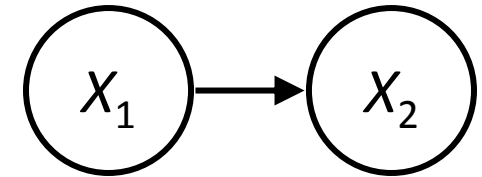
- Every time step, we start with current  $P(X \mid \text{evidence})$
- We update for time:

$$P(x_t | e_{1:t-1}) = \sum_{x_{t-1}} P(x_{t-1} | e_{1:t-1}) \cdot P(x_t | x_{t-1})$$

- We update for evidence:

$$P(x_t | e_{1:t}) \propto_X P(x_t | e_{1:t-1}) \cdot P(e_t | x_t)$$

- The forward algorithm does both at once (and doesn't normalize)
- Problem: space is  $|X|$  and time is  $|X|^2$  per time step



# Pacman – Sonar (P4)

